In this note we determine the automorphism groups of the symmetric groups S_n. For $n = 2$ this is very easy: we have $S_2 \cong \mathbf{Z}_2$ and hence $\mathrm{Aut}(S_2)$ is trivial. Therefore we suppose from now on that $n > 2$. The main result is Theorem 8.

For the convenience of the reader we first recall some basic properties of the groups S_n and the subgroups A_n of *even* permutations.

Lemma 1. *Let* $n > 2$.
(a) The center of S_n *is trivial;*
(b) For $n > 3$ *the center of* A_n *is trivial.*

Proof. (a) Let $\sigma \in Z(S_n)$. If $\sigma \neq \mathrm{id}$, then there exist two distinct $a, b \in \{1, 2, \ldots, n\}$ with $\sigma(a) = b$. Choose $c \in \{1, 2, \ldots, n\}$ with $c \neq a$ and $c \neq b$. Then $(b\,c)\sigma \neq \sigma(b\,c)$ because $(b\,c)\sigma$ maps a to c, while $\sigma(b\,c)$ maps a to b. This shows that $\sigma = \mathrm{id}$ and $Z(S_n)$ must be trivial, as required.
(b) Similarly, suppose that $\sigma \in Z(A_n)$ is non-trivial. Pick two distinct $a, b \in \{1, 2, \ldots, n\}$ with $\sigma(a) = b$ and choose two elements $c, d \in \{1, 2, \ldots, n\}$ different from a and b. Then $(b\,c\,d)\sigma \neq \sigma(b\,c\,d)$ because the two permutations map a to different elements.

Lemma 2. *Two elements of* S_n *are conjugate if and only if they have the same cycle type.*

Proof. For any $\sigma \in S_n$ and any $d \leq n$ we have

$$\sigma(1\,2\,\ldots\,d)\sigma^{-1} \;=\; (\sigma(1)\,\sigma(2)\,\ldots\,\sigma(d)).$$

This shows that any conjugate of a d-cycle is again a d-cycle. Since every permutation is a product of disjoint cycles, it follows that the cycle types of conjugate permutations are the same. In the other direction, let $\tau = (a_1 \ldots a_r)(a_{r+1} \ldots a_s) \ldots (a_l \ldots a_m)$ and $\tau' = (a'_1 \ldots a'_r)(a'_{r+1} \ldots a'_s) \ldots (a'_l \ldots a'_m)$ be two permutations having the same cycle type. Define $\sigma \in S_n$ by $\sigma(a_i) = a'_i$ for $i = 1, 2, \ldots, m$. Then

$$\begin{aligned}\sigma\tau\sigma^{-1} &= \sigma(a_1 \ldots a_r)\sigma^{-1}\sigma(a_{r+1} \ldots a_s)\sigma^{-1} \ldots \sigma(a_l \ldots a_m)\sigma^{-1},\\ &= (a'_1 \ldots a'_r)(a'_{r+1} \ldots a'_s) \ldots (a'_l \ldots a'_m),\\ &= \tau'.\end{aligned}$$

This shows that τ and τ' are conjugate, as required.

Lemma 3. *Let* $n > 2$.
(a) The group A_n *is generated by 3-cycles.*
(b) Any normal subgroup of A_n *that contains a 3-cycle, is equal to* A_n *itself.*

Proof. (a) The product $(1\,2)(2\,3)$ is equal to the 3-cycle $(1\,2\,3)$. The product of two disjoint 2-cycles $(a\,b)$ and $(c\,d)$ is equal to $(a\,b)(b\,c)(b\,c)(c\,d)$ and is hence a product of two 3-cycles. Since any element of A_n is a product of an *even* number of transpositions, it is therefore a product of 3-cycles.
(b) Let $N \subset A_n$ be a normal subgroup and suppose that $(1\,2\,3) \in N$. Let $\sigma' \in A_n$ be an arbitrary 3-cycle. Then $\sigma' = \tau(1\,2\,3)\tau^{-1}$ for some $\tau \in S_n$. If $\tau \in A_n$, then $\sigma' \in N$ and we are done. If not, then $\tau' = \tau(1\,2)$ is in A_n and $\sigma' = \tau'(1\,3\,2)\tau'^{-1}$ is once again in N.

Lemma 4. *The commutator subgroup of S_n is equal to A_n. For $n \geq 5$ the commutator subgroup of A_n is equal to A_n itself.*

Proof. Since S_n/A_n is commutative, the commutator subgroup S_n' is contained in A_n. Conversely, we have $(1\,2)(1\,3)(1\,2)^{-1}(1\,3)^{-1} = (1\,2\,3)$, showing that every 3-cycle is in S_n'. By Lemma 3 (a) the group A_n is generated by 3-cycles, so that $S_n' = A_n$ as required.

The identity

$$(1\,2\,3)(3\,4\,5)(1\,2\,3)^{-1}(3\,4\,5)^{-1} \;=\; (1\,4\,3).$$

shows that for $n \geq 5$ every 3-cycle is a commutator of A_n. This implies the second statement.

We remark that the group A_3 is abelian, so that its commutator subgroup is trivial. The group A_4 is not abelian. Its commutator subgroup is

$$V_4 \;=\; \{(1),\,(1\,2)(3\,4),\,(1\,3)(2\,4),\,(1\,4)(2\,3)\}.$$

Indeed, V_4 is normal and the quotient A_4/V_4 has order 3 and is hence abelian. It follows that $A_4' \subset V_4$. Equality follows from the identity $(1\,2\,3)(1\,2\,4)(1\,2\,3)^{-1}(1\,2\,4)^{-1} = (1\,2)(3\,4)$.

Proposition 5. *Let $n \geq 5$. Then the group A_n is simple, i.e. does not contain any proper normal subgroups. The only proper normal subgroup of S_n is A_n.*

Proof. Let $N \subset A_n$ be a non-trivial normal subgroup. We will show that N contains a 3-cycle. Then Lemma 3 (b) implies the required result.

Step 1. Suppose that N contains a permutation σ which is a product of disjoint cycles at least one of which has length $d \geq 4$. Then, up to renumbering, we have $\sigma = (1\,2\,\ldots\,d)\tau$ where τ leaves $\{1, 2, \ldots, d\}$ invariant. The permutation $\sigma^{-1}(1\,2\,3)\sigma(1\,2\,3)^{-1}$ is contained in N. One easily checks that it is equal to the 3-cycle $(1\,3\,d)$.

Step 2. This leaves us with the possibility that all permutations in N are products of disjoint cycles of length ≤ 3. Suppose that N contains a permutation σ admitting a 3-cycle. If it admits *only one* 3-cycle, then its square is a 3-cycle and we are done. If it contains *at least two* 3-cycles, we may assume that $\sigma = (1\,2\,3)(4\,5\,6)\tau$ where τ leaves $\{1, 2, \ldots, 6\}$ invariant. Then $\sigma^{-1}(1\,2\,4)\sigma(1\,2\,4)^{-1}$ is contained in N. One easily checks that it is equal to $(1\,4\,2\,3\,6)$ and we are done by Step 1.

Step 3. This leaves us with the possibility that all permutations in N are products of disjoint transpositions. Let $\sigma \in N$ be a non-trivial element. Since σ is even, it is a product of at least two transpositions and we may assume that $\sigma = (1\,2)(3\,4)\tau$, where τ leaves $\{1, 2, 3, 4\}$ invariant. Then $\sigma(1\,2\,3)\sigma(1\,2\,3)^{-1} = (1\,3)(2\,4)$ is in N. Since $n \geq 5$ the permutation $(1\,3)(2\,4)(1\,3\,5)(1\,3)(2\,4)(1\,3\,5)^{-1}$ is in N. It is equal to the 3-cycle $(1\,3\,5)$ and we are done.

To prove the second statement of the Proposition, let N be a proper normal subgroup of S_n. Then $N \cap A_n$ is a normal subgroup of A_n. So either $N \subset A_n$ in which case $N = \{1\}$ or $N = A_n$ or we have $N \cap A_n = \{1\}$. In the latter case $\#N \leq 2$ and hence $N \subset Z(S_n)$. Lemma 1 implies then that $N = \{1\}$. This proves the proposition.

We remark that the possibility that arises in Step 3 of the proof of Lemma 5, actually occurs for $n = 4$. In that case the group V_4 mentioned above is a normal subgroup of A_4. Its elements are products of disjoint transpositions.

Corollary 6. *For $n \geq 5$, the proper subgroups of A_n have index at least n.*

Proof. Let $H \subset A_n$ be a subgroup of index m. Translation of the left cosets of H gives rise to a non-trivial homomorphism $A_n \longrightarrow S_m$. By Lemma 4 the group A_n admits no proper normal subgroups, so that the map is injective. This implies $\frac{1}{2}n! \leq m!$, which can only happen when $n \leq m$ as required.

Lemma 7. *Let $n > 2$ and let $F : S_n \longrightarrow S_n$ be an automorphism mapping transpositions to transpositions. Then F is an inner automorphism.*

Proof. The product of two distinct transpositions has order 2 when the transpositions are disjoint, while it is a 3-cycle and thus has order 3 when they are not. Therefore any automorphism of S_n maps pairs of disjoint transpositions to pairs of disjoint transpositions.

Let $F(1\,2) = (a\,b)$. Let $x \in \{1, 2, \ldots, n\}$ be different from 1 or 2. Since $(1\,2)(1\,x)$ is a 3-cycle, so is $F(1\,2)F(1\,x) = (a\,b)F(1\,x)$. It follows that the 2-cycle $F(1\,x)$ moves either a or b. Switching a and b if necessary, we may assume that it moves a, so that $F(1\,x) = (a\,c)$ for some c different from a and b.

Claim. For every $y \neq 1$ we have $F(1\,y) = (a\,d)$ for some $d \in \{1, 2, \ldots, n\}$ different from a.

Proof of the claim. This is clear when $y = 2$ or $y = x$, so we may assume $y \neq 2, x$. Since both permutations $(1\,y)(1\,2)$ and $(1\,y)(1\,x)$ are 3-cycles, so are their images under F. We have $F(1\,2) = (a\,b)$ and $F(1\,x) = (a\,c)$. Therefore, if $F(1\,y)$ is not moving a, then it must move both b and c. Since $F(1\,y)$ is a transposition, this means that $F(1\,y) = (b\,c)$. Applying F^{-1} to the relation

$$(a\,b)(a\,c)(b\,c) \;=\; (a\,c),$$

we find

$$(1\,2)(1\,x)(1\,y) \;=\; (1\,x).$$

Since $y \notin \{1, 2, x\}$, the permutation on the left maps 1 to y. Since $x \neq y$, this is absurd and we conclude that $F(1\,y)$ actually moves a so that $F(1\,y) = (a\,d)$ for some d as required.

Define the permutation $\sigma \in S_n$ by putting $\sigma(1) = a$ and for every $y \neq 1$ put $\sigma(y) = d$, where d is the unique element for which $F(1\,y) = (a\,d)$. Its existence is guaranteed by the claim. Let $s : S_n \longrightarrow S_n$ denote the conjugation by σ map. For every y we have

$$s^{-1}F(1\,y) \;=\; s^{-1}(a\,d) \;= \sigma^{-1}(a\,d)\sigma = (1\,y).$$

In other words, $s^{-1}F$ fixes all transpositions of the form $(1\,y)$. Since $(y\,z) = (1\,z)(1\,y)(1\,z)$, the group S_n is generated by these transpositions. Therefore $s^{-1}F$ fixes every element of S_n, so that $F = s$. This proves the lemma.

Consider the homomorphism

$$S_n \;\longrightarrow\; \mathrm{Aut}(S_n)$$

that maps $\sigma \in S_n$ to the automorphism given by conjugation by σ. Its image is the subgroup of *inner automorphisms* of $\mathrm{Aut}(S_n)$. The kernel is precisely the center of S_n. Lemma 1 implies therefore that it is trivial. Therefore we can identify S_n with its image in $\mathrm{Aut}(S_n)$. The main result of this note if the following

Theorem 8. *Let $n > 2$. We have $\mathrm{Aut}(S_n) = S_n$ except when $n = 6$. In the exceptional case we have $[\mathrm{Aut}(S_6) : S_6] = 2$.*

Proof. The involutions (i.e. elements of order 2) of S_n are precisely the products of disjoint transpositions. For each k with $1 \leq k \leq n/2$, the set of products of k disjoint transpositions make up a conjugacy class C_k of S_n. Any automorphism of S_n maps involutions to involutions. Moreover, any automorphism F of S_n has the property that when $\sigma, \tau \in S_n$ are conjugate, so are $F(\sigma)$ and $F(\tau)$. Therefore an automorphism of S_n necessarily *permutes* the conjugacy classes C_k. We have

$$\#C_k \quad = \quad \frac{1}{k!}\binom{n}{2}\binom{n-2}{2}\ldots\binom{n-2(k-1)}{2}.$$

Let $n \neq 6$. Then an application of Lemma 9 below shows that $\#C_1 \neq \#C_k$ for any $k \neq 1$. It follows that an automorphism $F : S_n \longrightarrow S_n$ necessarily maps C_1 to itself. In other words, F maps transpositions to transpositions. Lemma 7 implies then that F is an inner automorphism, as required.

When $n = 6$, of number of transpositions is 15. This is the same as the number $\frac{1}{3!}\binom{6}{2}\binom{6}{4}$ of involutions of cycle type $(1\,2)(3\,4)(5\,6)$. On the other hand, there are 45 involutions of cycle type $(1\,2)(3\,4)$. In other words, we have $\#C_1 = \#C_3 = 15$, while $\#C_2 = 45$. See Table 12 below. Therefore, any automorphism $F : S_6 \longrightarrow S_6$ either preserves the transpositions and is by Lemma 2 an *inner* automorphism or it switches the conjugacy classes C_1 and C_3 and is *not* an inner automorphism. It follows that the composition of any two non-inner automorphisms preserves C_1 and is interior. This shows that $[\mathrm{Aut}(S_6) : S_6] \leq 2$. Below we actually construct a non-inner automorphism of S_6, showing that the index is 2, as required.

Lemma 9. *The only solution $k, m \in \mathbf{Z}$ of the equation*

$$\binom{m}{2}\binom{m-2}{2}\ldots\binom{m-2(k-1)}{2} \quad = \quad (k+1)!$$

with $m \geq 3$ and $1 \leq k \leq m/2$, is given by $m = 4$ and $k = 2$.

Proof. The left hand side of the equation is equal to

$$\frac{m(m-1)\ldots(m+2-2k)(m+1-2k)}{2^k} \quad = \quad \frac{m!}{(m-2k)!2^k}.$$

Therefore the equation can be rewritten as

$$\binom{m}{2k} \quad = \quad \frac{(k+1)!2^k}{(2k)!}.$$

For $k = 1$ this becomes $m(m-1) = 4$, which has no solutions in $\mathbf{Z}$. For $k = 2$ we find $m(m-1)(m-2)(m-3) = 24$ whose only solutions in $\mathbf{Z}$ are $m = 4$ and $m = -1$. For $k \geq 3$

the right hand side of the equation is less than 1. On the other hand, since $1 \leq k \leq m/2$, the binomial coefficient $\binom{m}{2k}$ is a positive integer. This shows that there are no further solutions, as required.

Construction of an outer automorphism of S_6. Consider the symmetric group S_5. It contains 24 5-cycles and hence has six 5-Sylow subgroups. Since S_5 acts transitively on its 5-Sylow subgroups, we obtain a homomorphism $j : S_5 \longrightarrow S_6$ whose image has cardinality at least 6. By Lemma 4 the group S_5 has no non-trivial normal subgroups except A_5. Therefore j is *injective.*

Remark. *The homomorphism $j : S_5 \longrightarrow S_6$ preserves parity.*

Proof. By Corollary 6 the group A_6 does not admit any subgroups of index 3. Therefore the image of j is not contained in A_6 and the map $\varepsilon : S_5 \longrightarrow S_6 \longrightarrow \{\pm 1\}$ is surjective. By Prop. 5 the kernel can only be the normal subgroup A_5. Therefore the map ε agrees with usual sign $S_5 \longrightarrow \{\pm\}$.

Let H denote the image of j. It is isomorphic to S_5 and has index 6 inside S_6. Let X denote the set of left cosets of H. The group S_6 acts on X by left translation. This gives rise to a homomorphism

$$F : S_6 \;\longrightarrow\; S(X) \;\cong\; S_6,$$

which is injective, because S_6 contains no proper normal subgroups except A_6.

The homomorphism F is an outer automorphism of S_6. Indeed, suppose that $F(1\,2)$ is a transposition. Then it has fixed points. This means that $(1\,2)xH = xH$ for some coset xH. It follows that H contains the transposition $x^{-1}(1\,2)x$. Since the homomorphism $S_5 \longrightarrow S_6$ preserves parity, the permutation $\sigma \in S_5$ that is mapped to this transposition is *odd.* It follows that σ is a transposition that normalizes a 5-Sylow subgroup P of S_5.

We may assume that P is generated by $(1\,2\,3\,4\,5)$ and that σ fixes 1. Then

$$\sigma(1\,2\,3\,4\,5)\sigma^{-1} = (1\,\sigma(2)\,\sigma(3), \sigma(4)\,\sigma(5))$$

is equal to $(1\,2\,3\,4\,5)$ or its inverse. This implies that $\sigma = \mathrm{id}$ or $\sigma = (2\,5)(3\,4)$ respectively, contradicting the fact that σ is odd. We conclude that $F(1\,2)$ is *not* a transposition. Lemma 7 implies now that F is not an inner automorphism. This proves Theorem 2.

Indeed, as was explained above, the automorphsm F constructed above necessarily switches the transpositions and the involution with cycle type $(1\,2)(3\,4)(5\,6)$. In order to describe certian properties of the outer automorphisms of S_6, we consider the normalizer of a 5-Sylow subgroup P of S_5.

Lemma 10. *The normalizer of a 5-Sylow subgroup of S_5 has 20 elements.*

Proof. Suppose that P is generated by $(1\,2\,3\,4\,5)$. Then the 4-cycle $(2\,3\,5\,4)$ normalizes P. The group generated by $(1\,2\,3\,4\,5)$ and $(2\,3\,5\,4)$ is contained in $N(P)$. It has order 20 and is not contained in A_5. If $N(P)$ were strictly larger, then its intersection with A_5 would be a subgroup of A_5 of index 2 or 3. This is impossible by Corollary 6. The proves the Lemma.

Corollary 11. *Any outer automorphism F of S_6 switches the* 3*-cycles and permutations of type* $(1\,2\,3)(4\,5\,6)$ *and it switches the* 6*-cycles and the permutations of type* $(1\,2\,3)(4\,5)$.

Proof. If F were to map $(1\,2\,3)$ to a 3-cycle, the subgroup H that appears in the construction of the non-inner automorphism F above, contains a 3-cycle. It is the image of a 3-cycle in S_5. Since a 3-cycle has fixed points, the normalizer of a 5-Sylow subgroup of S_5 contains a permutation of order 3. This contradicts Lemma 10. Similarly, if F maps the conjugacy class of permutations of type $(1\,2\,3)(4\,5)$ to itself, then H contains a permutation of type $(1\,2\,3)(4\,5)$. Since such a permutation has a fixed point, this means that the normalizer of a 5-Sylow subgroup of S_5 contains an element of order 6, contradicting Lemma 5.

The conjugacy classes of the 4-cycles and of the permutations of type $(1\,2\,3\,4)(5\,6)$ both contain 90 elements. However, the automorphism F does *not* switch these conjugacy classes, because it preserves the characteristic subgroup A_6 of S_6. Therefore the signs of the permutations σ and $F(\sigma)$ are equal for all σ.

In conclusion, in the table below, any outer automorphism of S_6 switches the conjugacy classes (i) and (ii), it switches (vi) and (vii) and it switches (viii) and (ix). It preserves the other ones.

Table 12. Conjugacy classes of S_6.

conjugacy class	cycle type	order	sign	#	#
(i)	$(1\,2\,3\,4\,5\,6)$	6	$-$	$5!$	120
(ii)	$(1\,2\,3\,)(4\,5)$	6	$-$	$6 \cdot 2 \cdot \binom{5}{2}$	120
(iii)	$(1\,2\,3\,4\,5)$	5	$+$	$\frac{1}{5}6!$	144
(iv)	$(1\,2\,3\,4)$	4	$-$	$\binom{6}{2}3!$	90
(v)	$(1\,2\,3\,4)(5\,6)$	4	$+$	$\binom{6}{2}3!$	90
(vi)	$(1\,2\,3)$	3	$+$	$2\binom{6}{3}$	40
(vii)	$(1\,2\,3)(4\,5\,6)$	3	$+$	$\frac{1}{2} \cdot 4 \cdot \binom{6}{3}$	40
(viii)	$(1\,2)$	2	$-$	$\binom{6}{2}$	15
(ix)	$(1\,2)(3\,4)(5\,6)$	2	$-$	$\frac{1}{3!}\binom{6}{2}\binom{6}{4}$	15
(x)	$(1\,2)(3\,4)$	2	$+$	$\frac{1}{2}\binom{6}{2}\binom{6}{4}$	45
(xi)	(1)	1	$+$	1	1