EXERCISES "LOW-RANK OPTIMIZATION METHODS"

Problem 1: Let H be a real Hilbert space, A a symmetric positive semidefinite linear operator on H (that is $\langle x, Ax\rangle = \langle Ax, x\rangle \geq 0$ for all x) and $b \in H$. Show that the function $f(x) = \frac{1}{2}\langle x, Ax\rangle - \langle x, b\rangle$ on H is convex and determine the gradient. Does f necessarily has a global minimum?

Problem 2: Let X_1 be the best rank-one approximation of a matrix $X \neq 0$ in a unitarily invariant norm. Show that $\langle \frac{X}{\|X\|_F}, \frac{X_1}{\|X_1\|_F}\rangle_F \geq \frac{1}{\sqrt{\operatorname{rank}(X)}}$.

Problem 3. Let $F(u,v)$ be a smooth function on $W = \mathbb{R}^N \times \mathbb{R}^M$. Let (u^*, v^*) be a local minimum of F. Let $\nabla^2 F(u^*, v^*) = \begin{bmatrix} A_{uu} & A_{uv} \\ A_{vu} & A_{vv} \end{bmatrix}$ be a block partition of the Hessian corresponding to the block variables u and v. Assume the diagonal blocks A_{uu} and A_{vv} are positive definite.

i) Show that there are neighborhoods W_u^1, W_u^2, W_v^1, W_v^2 of (u^*, v^*) in which two smooth functions $S_u \colon W_u^1 \to W_u^2$ and $S_v \colon W_v^1 \to W_v^2$ can be defined through
$$\nabla_u F(S_u(u,v), v) = 0, \quad \nabla_v F(u, S_v(u,v))) = 0.$$
(These two operators correspond to the single steps in alternating optimization). Moreover, we can have $W_u^2 \subseteq W_v^1$.
Hint: implicit function theorem.

ii) Prove that the function $S = S_2 \circ S_1$ has the derivative
$$S'(u^*, v^*) = -(L+D)^{-1}R = -\begin{bmatrix} A_{uu} & 0 \\ A_{vu} & A_{vv} \end{bmatrix}^{-1} \begin{bmatrix} 0 & A_{uv} \\ 0 & 0 \end{bmatrix}.$$
Hint: Consider the function $\phi((u,v),(\tilde{u},\tilde{v})) = (\nabla_u F(u,\tilde{v}), \nabla_v F(u,v))$ and $G(u,v) = \phi(S(u,v),(u,v))$. Then $G(u,v) = 0$ on W_u^1. Differentiate this equation!

Problem 4. Show that the hard thresholding operator $\mathcal{T}_k$ which maps a matrix to (one of) its best rank-k approximations in Frobenius norm is not a contraction with respect to this norm.

Problem 5. Consider a modified soft thresholding operator $\tilde{\mathcal{S}}_\epsilon(X) = \mathcal{S}_\epsilon(X)/\|\mathcal{S}_\epsilon(X)\|_F$, where $\mathcal{S}_\epsilon(U\Sigma V^T) = U(\Sigma - \epsilon I)_+ V^T$ as in the lecture. What are the fixed points of such an operator? Is it a contraction on the Frobenius norm unit sphere?

Problem 6: Let $M \subseteq \mathbb{R}^n$ be an embedded submanifold, $x \in M$ and R_x a map on $T_x M$ such that $R_x(\xi) \in M$ for all $\xi \in T_x M$ and
$$\|x + \xi - R_x(\xi)\| = \min_{y \in M} \|x + \xi - y\|$$
(at least for $\|\xi\|$ small enough). Show that is differentiable in a neighborhood of zero and $R_x'(0)$ equals the identity on $T_x M$. So the metric projection is a retraction.
If $M = M_k$ is the manifold of fixed rank-k matrices and $X \in M_k$, what is the largest neighborhood of zero in $T_X M_k$ for which this retraction is well defined?

Some References

Horn, Johnson: Matrix Analysis, Cambridge University Press, 2012.

Ortega, Rheinboldt: Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, 1970.

Absil, Mahony, Sepulchre: Optimization Algorithms on Matrix Manifolds, Princeton, 2007.