- Given $A \in \mathbb{C}^{n\times n}$, prove the following implication:

$$\underline{z}^H A \underline{z} \in \mathbb{R},\ \forall \underline{z} \in \mathbb{C}^n \Rightarrow A = A^H.$$

Thus, the positive definite matrices (i.e. the matrices $A$ such that $\underline{z}^H A \underline{z} > 0,\ \forall \underline{z} \in \mathbb{C}^n,\ \underline{z} \neq \underline{0}$) must be Hermitian.

- Set $\Pi_{-1} = \begin{pmatrix} 0 & 1 & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & 1 \\ -1 & 0 & \cdots & 0 \end{pmatrix}$ ($n$ rows) and $C_{-1}(\underline{\alpha}) = \sum_{k=0}^{n-1} \alpha_k (\Pi_{-1})^k$, $\underline{\alpha} \in \mathbb{C}^n$.

Find a diagonal unitary matrix $D$ such that, if $U = \frac{1}{\sqrt{n}} D W$, $W_{r,s} = (\omega_n)^{(r-1)(s-1)}$, $1 \leq r,s \leq n$ ($\omega_n \in \mathbb{C}$, $(\omega_n)^n = 1$, $(\omega_n)^k \neq 1$, $0 < k < n$), then $C_{-1}(\underline{\alpha}) = \sqrt{n}\, U\, d(U^T \underline{\alpha})\, U^H$.
(Note that $W^H W = nI$, thus $U$ is unitary).

- Given $A \in \mathbb{C}^{n\times n}$ set $\mathcal{F}_A = \left\{ \frac{\underline{x}^H A \underline{x}}{\underline{x}^H \underline{x}} : \underline{x} \in \mathbb{C}^n,\ \underline{x} \neq \underline{0} \right\}$.
Prove that $\mathcal{F}_A$ is a bounded convex subset of $\mathbb{C}$.

- Let $J_1, J_2, \ldots, J_m \in \mathbb{C}^{n\times n}$ be linearly independent.
Prove that the $m \times m$ matrix $B$, $B_{r,s} = (J_r, J_s)_F$, $1 \leq r,s \leq m$, is positive definite.
($(X,Y)_F = \sum_{i,j=1}^{n} \bar{x}_{ij} y_{ij}$, $\|X\|_F = (X,X)_F^{1/2}$)

- Let $\mathcal{L}$ be a subspace of $\mathbb{C}^{n\times n}$ and let $\mathcal{L}_A$ denote the unique element in $\mathcal{L}$ which makes $\|X - A\|_F$, $X \in \mathcal{L}$, minimum. Assume $\mathcal{L}_A = (\mathcal{L}_A)^H$ whenever $A = A^H$ (or, equivalently, $\mathcal{L}_A = -(\mathcal{L}_A)^H$ whenever $A = -A^H$).
Prove that $\mathcal{L}^H \subseteq \mathcal{L}$.

- Given $\underline{p}, \underline{q} \in \mathbb{C}^t$ prove that

$$\|\underline{p} - x\underline{q}\|_2 \geq \|\underline{p} - \frac{\underline{q}^H\underline{p}}{\underline{q}^H\underline{q}}\underline{q}\|_2, \quad \forall x \in \mathbb{C},$$

with the equality verified if and only if $x = \frac{\underline{q}^H\underline{p}}{\underline{q}^H\underline{q}}$.

CONSEQUENCE: Given arbitrary complex numbers $p_k$, $k=1,\dots,t$, we have $\sum_{k=1}^{t} |p_k - x|^2 \geq \sum_{k=1}^{t} \left| p_k - \frac{\sum_{s=1}^{t} p_s}{t} \right|^2$ with the equality verified iff $x = \sum_{s=1}^{t} p_s / t$

- Let $L$ be a normal $n \times n$ matrix, and let $Q_L$ be a unitary $n \times n$ matrix such that

$$Q_L^H L Q_L = \begin{pmatrix} \lambda_1(L) & & 0 \\ & \lambda_2(L) & \\ & & \ddots \\ 0 & & \lambda_n(L) \end{pmatrix}.$$

Prove that, for any polynomial $p$, we have $Q_L^H p(L) Q_L = \begin{pmatrix} p(\lambda_1(L)) & & 0 \\ & \ddots & \\ 0 & & p(\lambda_n(L)) \end{pmatrix}$.

- Set $\mathcal{L} = \{(t_{i-j})_{i,j=1}^{n} : t_k \in \mathbb{C}\}$.
Observe that $\mathcal{L}$ is closed under conjugate transposition and $I \in \mathcal{L}$, but positive definiteness is not preserved projecting on $\mathcal{L}$.

- Set $\mathcal{L} = \text{Span}\{L, M\}$, $L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $M = \frac{1}{\sqrt{10}}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, and $A = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}$. Observe that $\mathcal{L}_A$ is not positive definite.
Thus $\mathcal{L}^H \subseteq \mathcal{L}$ and $\exists M \in \mathcal{L}$, $M$ pd, are necessary but not sufficient conditions for pd preserved.

- Set

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \varepsilon \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \varepsilon > 0.$$

i) Observe that $A$ is not normal.
ii) Prove that there exists $\varepsilon^* > 0$ such that $\forall \varepsilon \in (0, \varepsilon^*)$ the sets $\mathcal{F}_A$ and $[\mathcal{O}(A)]$ = "SMALLEST CONVEX POLYGON INCLUDING $\mathcal{O}(A)$" coincide.