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THE ANTITRIANGULAR FACTORIZATION OF SYMMETRIC MATRICES*

NICOLA MASTRONARDI† AND PAUL VAN DOOREN‡

Abstract. Indefinite symmetric matrices occur in many applications, such as optimization, least squares problems, partial differential equations, and variational problems. In these applications one is often interested in computing a factorization of the indefinite matrix that puts into evidence the inertia of the matrix or possibly provides an estimate of its eigenvalues. In this paper we propose an algorithm that provides this information for any symmetric indefinite matrix by transforming it to a block antitriangular form using orthogonal similarity transformations. We also show that the algorithm is backward stable and has a complexity that is comparable to existing matrix decompositions for dense indefinite matrices.

Key words. indefinite matrix, saddle point problem, inertia, eigenvalue estimate

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1. Introduction.

Indefinite symmetric matrices occur in many applications, such as optimization, partial differential equations, and variational problems where they are, for instance, linked to a so-called saddle point problem. In these applications one is typically interested in computing a factorization of the indefinite matrix that puts into evidence the inertia of the matrix or possibly provides an estimate of its eigenvalues. We develop in this paper a new type of matrix factorization that provides exactly that type of information, and we present a backward stable algorithm, relying only on Givens and Householder transformations, for computing it.

In the literature, there are already several types of matrix factorizations of a symmetric matrix A that provide very similar information. The LDL^T factorization, where L is unit-lower triangular and D is a block-diagonal matrix with 1×1 and 2×2 diagonal blocks, determines immediately the inertia of A via its diagonal. The UTU^T decomposition, where U is orthogonal and T is tridiagonal, can be combined with Sturm sequences to give exactly the number of positive, negative, and zero eigenvalues of A. A third factorization is the LTL^T decomposition, where L is unit lower-triangular and T is tridiagonal, which can also be combined with Sturm sequences to determine the inertia of A.

The LDL^T factorization of an $n \times n$ symmetric matrix A is typically combined with certain pivoting strategies and requires $O(n^3)$ floating point operations. Algorithms for constructing such factorizations (Bunch and Parlett [5] and Bunch and Kaufman [6]) have been proved to be backward stable (see, e.g., [14]), provided they incorporate a good pivoting strategy. The pivoting strategy ensuring backward sta-

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†Istituto per le Applicazioni del Calcolo "M. Picone," Consiglio Nazionale delle Ricerche, sede di Bari, I-70126 Italy (n.mastronardi@ba.iac.cnr.it). The work of this author is partly supported by GNCS–INdAM project *Analisi di strutture nella ricostruzione di immagini e monumenti.*

‡Department of Mathematical Engineering, Catholic University of Louvain, Louvain-la-Neuve, B-1348 Belgium (paul.vandooren@uclouvain.be). The work of this author is partly supported by the Belgian Network DYSCO (Dynamical Systems, Control, and Optimization), funded by the Interuniversity Attraction Poles Programme, initiated by the Belgian State Science Policy Office. The scientific responsibility rests with its authors.

bility in the LTL^T decomposition is much simpler [1]. But in both cases the transformations are nonorthogonal, and therefore the central matrices D and T do not provide any useful information about the eigenvalues of A, besides their sign. The UTU^T decomposition, on the other hand, uses orthogonal transformations, and hence the eigenvalues of A are also those of T. This makes it possible to obtain additional estimates on the eigenvalues of A at low additional cost.

In this paper we present a new symmetric factorization $A = QMQ^T$, where Q is orthogonal and M is symmetric block lower antitriangular (which means that blocks above the main antidiagonal are zero) requiring $O(n^3)$ floating point operations. Since Q is orthogonal, the eigenvalues of A and M are the same. The sizes of the blocks of M are shown to yield the inertia of A, and when all the eigenvalues of A are located into two clusters, a good estimation of them can be obtained by computing the eigenvalues of a matrix formed by the main diagonal and the main antidiagonal of some blocks of M. Similar antisymmetric decompositions have appeared in the literature in the context of matrix pencils with particular types of symmetries [12, 21, 17, 16]. The problem of tracking the dominant eigenspace of a symmetric indefinite matrix, making use of the block antitriangular decomposition, has been considered in [20].

The paper is organized as follows. In section 2 it is shown that any symmetric matrix can be transformed into a block lower antitriangular matrix. Moreover, an algorithm for computing the factorization $A = QMQ^T$ is proposed. Some bounds and inequalities on the values on the main diagonal and antidiagonal of the computed block antitriangular matrix are also given. An algorithm for the rank-one updating/downdating of block antitriangular matrices is proposed in section 3. Numerical experiments, showing the properties of the proposed algorithm, are reported in section 4 followed by the conclusions.

2. The antitriangular decomposition. In this section, we describe the reduction of an arbitrary symmetric matrix to a block lower antitriangular form via orthogonal similarity transformations. We will say that a matrix $A \in \mathbb{R}^{n\times n}$ is lower antitriangular if $A(i,j) = 0,\ i+j \le n$. The *inertia* of a symmetric matrix $A \in \mathbb{R}^{n\times n}$ is the triple $\text{Inertia}(A) = (n_-, n_0, n_+)$, where n_-, n_0, and n_+ are the number of negative, zero, and positive eigenvalues of A, respectively, and $n_- + n_0 + n_+ = n$.

A symmetric indefinite matrix $A \in \mathbb{R}^{n\times n}$ is in block antitriangular form if

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{0} & Y^T \\ \mathbf{0} & X & Z^T \\ Y & Z & W \end{bmatrix}$$

with Y, X, and W square submatrices of different sizes, X and W symmetric, and Y antitriangular. Moreover, a symmetric indefinite matrix $A \in \mathbb{R}^{n\times n}$ with $\text{Inertia}(A) = (n_-, n_0, n_+)$, $n_1 = \min(n_-, n_+)$, and $n_2 = \max(n_-, n_+) - n_1$ is in proper block antitriangular form if

$$A = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y^T \\ \mathbf{0} & \mathbf{0} & X & Z^T \\ \mathbf{0} & Y & Z & W \end{bmatrix} \begin{matrix} \}n_0 \\ \}n_1 \\ \}n_2 \\ \}n_1 \end{matrix} \tag{2.1}$$

with $Z \in \mathbb{R}^{n_1\times n_2}$, $W \in \mathbb{R}^{n_1\times n_1}$ symmetric, $Y \in \mathbb{R}^{n_1\times n_1}$ nonsingular lower antitriangular, and $X \in \mathbb{R}^{n_2\times n_2}$ symmetric definite if $n_2 > 0$, i.e., $X = \varepsilon LL^T$ with

$$\varepsilon = \begin{cases} 1 & \text{if } n_+ > n_-, \\ -1 & \text{if } n_+ < n_- \end{cases}$$

and L lower triangular. Otherwise, ε is not defined. Hence, X is symmetric positive definite if $\varepsilon = 1$ and is symmetric negative definite if $\varepsilon = -1$.

Remark 1. Notice that some of the blocks of A may have zero dimension.

The identity matrix of order n is denoted by I_n, its columns, the unit vectors, are denoted by $\mathbf{e}_i$, $i = 1, \ldots, n$, and their size can vary depending on the context. If there is no ambiguity, different sequences of Givens rotations will be denoted by the same symbols, for instance, G_i, $i = 1, \ldots, k$. Submatrices are denoted by the colon notation of MATLAB: $A(i : j, k : l)$ denotes the submatrix of A formed by the intersection of rows i to j and columns k to l, and $A(:, k : l)$ denotes the columns of A from k to l. A null submatrix is denoted by $\mathbf{0}$, and its size can vary depending on the context. Moreover, the symbol "=" is also used to introduce new variables.

Here we show that any symmetric matrix can be transformed into a proper block antitriangular form by orthogonal similarity transformations.

THEOREM 2.1. *Let* $A \in \mathbb{R}^{n\times n}$ *be a symmetric indefinite matrix with* $\mathrm{Inertia}(A) = (n_-, n_0, n_+)$. *Let* $n_1 = \min(n_-, n_+)$, *and* $n_2 = \max(n_-, n_+) - n_1$. *Then, there exists an orthogonal matrix* $Q \in \mathbb{R}^{n\times n}$ *such that* $M = Q^T AQ$ *is in proper block antitriangular form,*

$$M = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y^T \\ \mathbf{0} & \mathbf{0} & X & Z^T \\ \mathbf{0} & Y & Z & W \end{bmatrix} \begin{matrix} \}n_0 \\ \}n_1 \\ \}n_2 \\ \}n_1 \end{matrix} .$$

Proof. The proof is based on the existence of a set of spaces associated with a symmetric matrix A with inertia (n_-, n_0, n_+) . Let us consider orthogonal bases U_0, U_+, U_-, and U_n; then the associated spaces $\mathcal{U}_0$, $\mathcal{U}_+$, $\mathcal{U}_-$, and $\mathcal{U}_n$ are called

- a nullspace of A if $AU_0 = \mathbf{0}$; the nullspace of maximal dimension is unique and of dimension n_0,
- a nonnegative subspace of A if $U_+^T AU_+$ is positive semidefinite; its maximal dimension is $n_0 + n_+$,
- a nonpositive subspace of A if $U_-^T AU_-$ is negative semidefinite; its maximal dimension is $n_0 + n_-$,
- a neutral subspace of A if $U_n^T AU_n = \mathbf{0}$; its maximal dimension is $n_0 + \min(n_+, n_-)$.

The properties of neutral, positive, and negative spaces are proved in, e.g., [9] but can also easily be recovered from the Cauchy interlacing inequalities. Without loss of generality, we now assume that $n_+ \geq n_-$. (If not, one can just consider $-A$ instead of A.) Suppose we are given the nullspace $\mathcal{U}_0$ and two subspaces $\mathcal{U}_n$ and $\mathcal{U}_+$ of maximal dimension; then $\mathcal{U}_0 \subseteq \mathcal{U}_n \subseteq \mathcal{U}_+$ since otherwise $\mathcal{U}_n$ and $\mathcal{U}_+$ would not be of maximal dimension. We then construct an orthogonal matrix $Q = [Q_1|Q_2|Q_3|Q_4]$, where Q_1 spans $\mathcal{U}_0$, Q_2 spans the orthogonal complement of $\mathcal{U}_0$ in $\mathcal{U}_n$, Q_3 spans the orthogonal complement of $\mathcal{U}_n$ in $\mathcal{U}_+$, and Q_4 spans the orthogonal complement of $\mathcal{U}_+$. Let us denote by $K \in \mathbb{R}^{n_2\times n_1}$ the block $(3, 2)$ of M. We observe that $K = \mathbf{0}$ because otherwise

$$[Q_1\ Q_2\ Q_3]^T A[Q_1\ Q_2\ Q_3] = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & K^T \\ \mathbf{0} & K & X \end{bmatrix}$$

is an indefinite matrix, contradicting the fact that $[Q_1\ Q_2\ Q_3]$ spans the subspace $\mathcal{U}_+$. It then follows that the number of columns of the Q_i matrices are, respectively,

$n_0, n_-, (n_+ - n_-)$, and n_- and that M must be of the form given above with Y and X square nonsingular. In fact Y is full column rank; otherwise the dimension of $\mathcal{U}_0$ is greater than n_0. Moreover, the matrix Y is also full row rank; otherwise the dimension of $\mathcal{U}_+$ is greater than $n_0 + n_1 + n_2$. Observe that X is nonsingular; otherwise the dimension of $\mathcal{U}_n$ would be greater than $n_0 + n_1$. The matrix Y can be reduced to antitriangular form by a QR-like factorization without changing the above subspaces. □

The fact that the block antitriangular structure of M determines its inertia has been proved in [10].

2.1. Reduction to antitriangular form. In this section it is shown how a symmetric matrix can be transformed into a block antitriangular one. The algorithm for constructing the form is recursive and is also a proof of the existence of the form. We suppose that we start with $A^{(k)} = A(1:k, 1:k)$, $1 \le k < n$, which is in this form, i.e., $A^{(k)} = Q^{(k)} M^{(k)} {Q^{(k)}}^T$ with $Q^{(k)} \in \mathbb{R}^{k \times k}$ orthogonal and $M^{(k)} \in \mathbb{R}^{k \times k}$ proper block antitriangular. Notice that such a form is trivial to obtain for $k = 1$. We then show how to transform $A^{(k+1)} = A(1:k+1, 1:k+1)$ into a proper block antitriangular form by updating orthogonal transformations. Let Inertia$(A^{(k)}) = (k_-, k_0, k_+)$, $k_- + k_0 + k_+ = k$, and let $k_1 = \min(k_-, k_+)$, $k_2 = \max(k_-, k_+) - k_1$. Let $M^{(k)}$ be the proper block antitriangular form of $A^{(k)}$, i.e.,

$$M^{(k)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y^T \\ \mathbf{0} & \mathbf{0} & X & Z^T \\ \mathbf{0} & Y & Z & W \end{bmatrix} \begin{matrix} \}k_0 \\ \}k_1 \\ \}k_2 \\ \}k_1 \end{matrix}$$

with $Y \in \mathbb{R}^{k_1 \times k_1}$ nonsingular lower antitriangular, $X \in \mathbb{R}^{k_2 \times k_2}$ symmetric definite, $Z \in \mathbb{R}^{k_1 \times k_2}$, and $W \in \mathbb{R}^{k_1 \times k_1}$ symmetric. Notice that some of the blocks may have zero dimension. Without loss of generality, let us assume that $k_2 > 0$ and X is positive definite with Cholesky factorization $X = LL^T$. Let $\mathbf{a} = A(1:k, k+1)$ and $\gamma = a_{k+1,k+1}$. Partition

$$\tilde{\mathbf{a}} = {Q^{(k)}}^T \mathbf{a} = \begin{bmatrix} \tilde{\mathbf{a}}_1 \\ \tilde{\mathbf{a}}_2 \\ \tilde{\mathbf{a}}_3 \\ \tilde{\mathbf{a}}_4 \end{bmatrix} \begin{matrix} \}k_0 \\ \}k_1 \\ \}k_2 \\ \}k_1 \end{matrix} . \tag{2.2}$$

Let $\tilde{Q}^{(k+1)} = \begin{bmatrix} Q^{(k)} & \\ & 1 \end{bmatrix}$. Since $A^{(k+1)} = \begin{bmatrix} A^{(k)} & \mathbf{a} \\ \mathbf{a}^T & \gamma \end{bmatrix}$,

$$\tilde{M}^{(k+1)} = {\tilde{Q}^{(k+1)}}^T A^{(k+1)} \tilde{Q}^{(k+1)} = \begin{bmatrix} M^{(k)} & \tilde{\mathbf{a}} \\ \tilde{\mathbf{a}}^T & \gamma \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \tilde{\mathbf{a}}_1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y^T & \tilde{\mathbf{a}}_2 \\ \mathbf{0} & \mathbf{0} & X & Z^T & \tilde{\mathbf{a}}_3 \\ \mathbf{0} & Y & Z & W & \tilde{\mathbf{a}}_4 \\ \tilde{\mathbf{a}}_1^T & \tilde{\mathbf{a}}_2^T & \tilde{\mathbf{a}}_3^T & \tilde{\mathbf{a}}_4^T & \gamma \end{bmatrix}.$$

We now show how to update the orthogonal similarity transformations in order to reduce $\tilde{M}^{(k+1)}$ to proper block antitriangular form, distinguishing the following 3 cases.

Case a. $\|[\tilde{\mathbf{a}}^T,\ \gamma]\|_2 = 0$.
Let

$$P = \begin{bmatrix} 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & 0 \\ & \ddots & \ddots & \ddots \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(k+1)\times(k+1)}$$

be a circulant matrix. Then $M_a^{(k+1)} = P\tilde{M}^{(k+1)}P^T$ is in the required block antitriangular form. So, for this case, let $Q_a^{(k+1)} = \tilde{Q}^{(k+1)}P^T$. Then $M_a^{(k+1)}$ is in proper block antitriangular form,

$$M_a^{(k+1)} = Q_a^{(k+1)^T} A^{(k+1)} Q_a^{(k+1)},$$

with $\mathrm{Inertia}(A^{(k+1)}) = (k_-, k_0+1, k_+)$.

Case b.[1] $\|\tilde{\mathbf{a}}_1\|_2 > 0$.
Let us consider the Householder matrix $H \in \mathbb{R}^{k_0\times k_0}$ such that $H\tilde{\mathbf{a}}_1 = \theta\mathbf{e}_{k_0}$. Let

$$Q_b^{(k+1)} = \tilde{Q}^{(k+1)} \begin{bmatrix} H^T & \\ & I_{k+1-k_0} \end{bmatrix}. \tag{2.3}$$

Then

$$M_b^{(k+1)} = Q_b^{(k+1)^T} A^{(k+1)} Q_b^{(k+1)}$$

is in proper block antitriangular form,

$$M_b^{(k+1)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y_b^T \\ \mathbf{0} & \mathbf{0} & X & Z_b^T \\ \mathbf{0} & Y_b & Z_b & W_b \end{bmatrix} \begin{matrix} \}k_0-1 \\ \}k_1+1 \\ \}k_2 \\ \}k_1+1 \end{matrix},$$

where

$$Y_b = \begin{bmatrix} \mathbf{0} & Y \\ \theta & \tilde{\mathbf{a}}_2^T \end{bmatrix},\ Z_b = \begin{bmatrix} Z \\ \tilde{\mathbf{a}}_3^T \end{bmatrix},\ W_b = \begin{bmatrix} W & \tilde{\mathbf{a}}_4 \\ \tilde{\mathbf{a}}_4^T & \gamma \end{bmatrix}.$$

Moreover, $\mathrm{Inertia}(A^{(k+1)}) = (k_-+1, k_0-1, k_++1)$.

Case c. $\|\tilde{\mathbf{a}}_1\|_2 = 0$.
This subcase happens, for instance, when $M^{(k)}$ is nonsingular, i.e., $k_0 = 0$, and $\tilde{\mathbf{a}}_1 = [\ ]$. The first step is to annihilate the entries of the main antidiagonal of the submatrix $[\ Y^T \quad \tilde{\mathbf{a}}_2\ ]^T$ by means of $\Gamma_1 \in \mathbb{R}^{(k_1+1)\times(k_1+1)}$, the product of k_1 Givens matrices such that

$$\begin{bmatrix} \mathbf{0} \\ Y_c \end{bmatrix} = \Gamma_1 \begin{bmatrix} Y \\ \tilde{\mathbf{a}}_2^T \end{bmatrix} \tag{2.4}$$

with Y_c nonsingular lower antitriangular of order k_1. Let

$$\begin{bmatrix} \mathbf{v}^T \\ Z_c \end{bmatrix} = \Gamma_1 \begin{bmatrix} Z \\ \tilde{\mathbf{a}}_3^T \end{bmatrix} \tag{2.5}$$

[1]From a computational point of view, all the binary comparisons with 0 must be replaced by comparisons with a user defined tolerance τ.

and

$$\left[\begin{array}{cc} \hat{\gamma} & \mathbf{w}^T \\ \mathbf{w} & W_c \end{array}\right] = \Gamma_1 \left[\begin{array}{cc} W & \tilde{\mathbf{a}}_4 \\ \tilde{\mathbf{a}}_4^T & \gamma \end{array}\right] \Gamma_1^T \tag{2.6}$$

with $\mathbf{v} \in \mathbb{R}^{k_2}$, $\mathbf{w} \in \mathbb{R}^{k_1}$. Moreover, let

$$Q_c^{(k+1)} = \tilde{Q}^{(k+1)} \left[\begin{array}{cc} I_{k-k_1} & \\ & \Gamma_1^T \end{array}\right]. \tag{2.7}$$

Then,

$$M_c^{(k+1)} = Q_c^{(k+1)^T} A^{(k+1)} Q_c^{(k+1)} = \left[\begin{array}{ccccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & Y_c^T \\ \mathbf{0} & \mathbf{0} & X & \mathbf{v} & Z_c^T \\ \mathbf{0} & \mathbf{0} & \mathbf{v}^T & \hat{\gamma} & \mathbf{w}^T \\ \mathbf{0} & Y_c & Z_c & \mathbf{w} & W_c \end{array}\right] \begin{array}{l} \}k_0 \\ \}k_1 \\ \}k_2 \\ \}1 \\ \}k_1 \end{array}. \tag{2.8}$$

Let us define the submatrix

$$X_1 = \left[\begin{array}{cc} X & \mathbf{v} \\ \mathbf{v}^T & \hat{\gamma} \end{array}\right]. \tag{2.9}$$

Then the next step of this subcase is to check the definiteness of X_1. Let $\Gamma_2 \in \mathbb{R}^{k_2 \times k_2}$ be the product of $k_2 - 1$ Givens rotations such that $\Gamma_2 \mathbf{v} = \alpha \mathbf{e}_{k_2}$ with $\alpha \in \mathbb{R}$. Then

$$\begin{aligned} X_2 = \left[\begin{array}{cc} \Gamma_2 & \\ & 1 \end{array}\right] X_1 \left[\begin{array}{cc} \Gamma_2^T & \\ & 1 \end{array}\right] &= \left[\begin{array}{cc} \Gamma_2 L \Gamma_3^T \Gamma_3 L^T \Gamma_2^T & \alpha \mathbf{e}_{k_2} \\ \alpha \mathbf{e}_{k_2}^T & \hat{\gamma} \end{array}\right] \\ &= \left[\begin{array}{cc} L_1 L_1^T & \alpha \mathbf{e}_{k_2} \\ \alpha \mathbf{e}_{k_2}^T & \hat{\gamma} \end{array}\right], \end{aligned} \tag{2.10}$$

where $\Gamma_3 \in \mathbb{R}^{k_2 \times k_2}$ is a sequence of $k_2 - 1$ "inner" Givens rotations chosen such that $L_1 = \Gamma_2 L \Gamma_3^T$ is nonsingular lower triangular. Let us decompose L_1 as

$$L_1 = \left[\begin{array}{cc} L_2 & \\ \mathbf{l}_2^T & \beta \end{array}\right]$$

with $L_2 \in \mathbb{R}^{(k_2-1) \times (k_2-1)}$ lower triangular, $\mathbf{l}_2 \in \mathbb{R}^{k_2-1}$, and $\beta \in \mathbb{R}$. It turns out that

$$X_2 = \left[\begin{array}{c} L_2 \\ \mathbf{l}_2^T \\ \mathbf{0} \end{array}\right] \left[\begin{array}{ccc} L_2^T & \mathbf{l}_2 & \mathbf{0} \end{array}\right] + \left[\begin{array}{c|cc} & & \\ \hline & \beta^2 & \alpha \\ & \alpha & \hat{\gamma} \end{array}\right].$$

Notice that $\beta \neq 0$. Since the Schur complement of $L_1 L_1^T$ in X_2 is $\hat{\gamma} - \alpha^2/\beta^2$, X_2 is positive definite, singular, or indefinite if the 2×2 matrix $T = \left[\begin{smallmatrix} \beta^2 & \alpha \\ \alpha & \hat{\gamma} \end{smallmatrix}\right]$ is positive definite, singular, or indefinite, respectively. If we define

$$Q_{c_1}^{(k+1)} = Q_c^{(k+1)} \left[\begin{array}{ccc} I_{k_0+k_1} & & \\ & \Gamma_2^T & \\ & & I_{k_1+1} \end{array}\right] \tag{2.11}$$

and

$$Z_{c_1} = \left[\begin{array}{cc} Z_c & \mathbf{w} \end{array}\right] \left[\begin{array}{cc} \Gamma_2^T & \\ & 1 \end{array}\right], \tag{2.12}$$

then we have to distinguish the following cases.

c.1. The matrix T is symmetric positive definite with

$$\begin{bmatrix} l_{1,1} & \\ l_{2,1} & l_{2,2} \end{bmatrix}$$

being its Cholesky factor. Therefore X_2 is symmetric positive definite (SPD) and so is X_1, and its Cholesky factor is given by

$$L_{c_1} = \begin{bmatrix} L_2 & & \\ \mathbf{l}_2^T & l_{1,1} & \\ \mathbf{0} & l_{2,1} & l_{2,2} \end{bmatrix}, \quad \text{i.e.,} \quad X_{c_1} = X_2 = L_{c_1} L_{c_1}^T.$$

Then,

$$M_{c_1}^{(k+1)} = Q_{c_1}^{(k+1)^T} A^{(k+1)} Q_{c_1}^{(k+1)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y_c^T \\ \mathbf{0} & \mathbf{0} & X_{c_1} & Z_{c_1}^T \\ \mathbf{0} & Y_c & Z_{c_1} & W_c \end{bmatrix} \begin{matrix} \}k_0 \\ \}k_1 \\ \}k_2+1 \\ \}k_1 \end{matrix}.$$

c.2. The matrix T is singular with eigenvalues $0 = \lambda_1 < \lambda_2$. Let Q_1 be the Givens rotation such that

$$Q_1 T Q_1^T = \begin{bmatrix} 0 & \\ & \lambda_2 \end{bmatrix}. \tag{2.13}$$

Then,

$$X_3 = \begin{bmatrix} I_{k_2-1} & \\ & Q_1 \end{bmatrix} X_2 \begin{bmatrix} I_{k_2-1} & \\ & Q_1^T \end{bmatrix} = \begin{bmatrix} L_2 \\ \mathbf{l}_3^T \\ \mathbf{l}_4^T \end{bmatrix} \begin{bmatrix} L_2^T & \mathbf{l}_3 & \mathbf{l}_4 \end{bmatrix} + \left[\begin{array}{c|cc} & & \\ \hline & 0 & \\ & & \lambda_2 \end{array}\right]$$

with

$$\begin{bmatrix} \mathbf{l}_3^T \\ \mathbf{l}_4^T \end{bmatrix} = Q_1 \begin{bmatrix} \mathbf{l}_2^T \\ \mathbf{0} \end{bmatrix}.$$

Let $\Gamma_4 \in \mathbb{R}^{k_2 \times k_2}$ be the product of $k_2 - 1$ Givens rotations such that

$$\begin{bmatrix} \mathbf{0} \\ L_3 \end{bmatrix} = \Gamma_4 \begin{bmatrix} L_2 \\ \mathbf{l}_3^T \end{bmatrix} \tag{2.14}$$

with $L_3 \in \mathbb{R}^{(k_2-1)\times(k_2-1)}$. Then

$$\begin{bmatrix} \Gamma_4 & \\ & 1 \end{bmatrix} X_3 \begin{bmatrix} \Gamma_4^T & \\ & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L_3 L_3^T & L_3 \mathbf{l}_4 \\ \mathbf{0} & \mathbf{l}_4^T L_3^T & \lambda_2 + \mathbf{l}_4^T \mathbf{l}_4 \end{bmatrix}. \tag{2.15}$$

Then $X_{c_2} = L_{c_2} L_{c_2}^T$ with $L_{c_2} = \begin{bmatrix} L_3 & \\ \mathbf{l}_4^T & \sqrt{\lambda_2} \end{bmatrix}$. Let

$$\tilde{Q}_{c_2}^{(k+1)} = Q_{c_1}^{(k+1)} \begin{bmatrix} I_{k_0+k_1+k_2-1} & & \\ & Q_1^T & \\ & & I_{k_1} \end{bmatrix} \begin{bmatrix} I_{k_0+k_1} & & \\ & \Gamma_4^T & \\ & & I_{k_1+1} \end{bmatrix}. \tag{2.16}$$

Then

$$\tilde{M}_{c_2}^{(k+1)} = \tilde{Q}_{c_2}^{(k+1)^T} A^{(k+1)} \tilde{Q}_{c_2}^{(k+1)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & Y_c^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{y}^T \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & X_{c_2} & Z_{c_2}^T \\ \mathbf{0} & Y_c & \mathbf{y} & Z_{c_2} & W_c \end{bmatrix} \begin{matrix} \}k_0 \\ \}k_1 \\ \}1 \\ \}k_2 \\ \}k_1 \end{matrix}$$

with

$$\begin{bmatrix} \mathbf{y} & Z_{c_2} \end{bmatrix} = Z_{c_1} \begin{bmatrix} I_{k_2-1} & \\ & Q_1^T \end{bmatrix} \begin{bmatrix} \Gamma_4^T & \\ & 1 \end{bmatrix}. \tag{2.17}$$

Let $\Gamma_5 \in \mathbb{R}^{(k_1+1)\times(k_1+1)}$ be the product of k_1 Givens rotations such that

$$\begin{bmatrix} \mathbf{0} & Y_{c_2} \end{bmatrix} = \begin{bmatrix} Y_c & \mathbf{y} \end{bmatrix} \Gamma_5^T \tag{2.18}$$

with $Y_{c_2} \in \mathbb{R}^{k_1 \times k_1}$ nonsingular lower antitriangular. Let

$$Q_{c_2}^{(k+1)} = \tilde{Q}_{c_2}^{(k+1)} \begin{bmatrix} I_{k_0} & & \\ & \Gamma_5 & \\ & & I_{k_1+k_2} \end{bmatrix}. \tag{2.19}$$

Then

$$M_{c_2}^{(k+1)} = Q_{c_2}^{(k+1)^T} A^{(k+1)} Q_{c_2}^{(k+1)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y_{c_2}^T \\ \mathbf{0} & \mathbf{0} & X_{c_2} & Z_{c_2}^T \\ \mathbf{0} & Y_{c_2} & Z_{c_2} & W_c \end{bmatrix} \begin{matrix} \}k_0+1 \\ \}k_1 \\ \}k_2 \\ \}k_1 \end{matrix}.$$

c.3. The matrix T is indefinite. Let Q_2 be the Givens rotation such that

$$Q_2 T Q_2^T = \begin{bmatrix} 0 & \xi_1 \\ \xi_1 & \xi_2 \end{bmatrix}. \tag{2.20}$$

Let

$$\begin{bmatrix} \mathbf{l}_5^T \\ \mathbf{l}_6^T \end{bmatrix} = Q_2 \begin{bmatrix} \mathbf{l}_2^T \\ \mathbf{0} \end{bmatrix}.$$

Then

$$\begin{aligned} X_3 &= \begin{bmatrix} I_{k_2-1} & \\ & Q_2 \end{bmatrix} X_2 \begin{bmatrix} I_{k_2-1} & \\ & Q_2^T \end{bmatrix} \\ &= \begin{bmatrix} L_2 \\ \mathbf{l}_5^T \\ \mathbf{l}_6^T \end{bmatrix} \begin{bmatrix} L_2^T & \mathbf{l}_5 & \mathbf{l}_6 \end{bmatrix} + \left[\begin{array}{c|cc} & & \\ \hline & 0 & \xi_1 \\ & \xi_1 & \xi_2 \end{array}\right]. \end{aligned}$$

Similar to the previous subcase, let $\Gamma_6 \in \mathbb{R}^{k_2 \times k_2}$ be the product of $k_2 - 1$ Givens rotations such that

$$\begin{bmatrix} \mathbf{0} \\ L_{c_3} \end{bmatrix} = \Gamma_6 \begin{bmatrix} L_2 \\ \mathbf{l}_5^T \end{bmatrix} \tag{2.21}$$

with $L_{c_3} \in \mathbb{R}^{(k_2-1)\times(k_2-1)}$. Then

$$\begin{bmatrix}\Gamma_6 & \\ & 1\end{bmatrix} X_3 \begin{bmatrix}\Gamma_6^T & \\ & 1\end{bmatrix} = \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & L_{c_3}L_{c_3}^T & L_{c_3}\mathbf{l}_6 \\ \mathbf{0} & \mathbf{l}_6^T L_{c_3}^T & \mathbf{l}_6^T\mathbf{l}_6\end{bmatrix} + \left[\begin{array}{c|c} & \mathbf{z} \\ \hline \mathbf{z}^T & \xi_2\end{array}\right], \tag{2.22}$$

where $\mathbf{z} = \xi_1\Gamma_6\mathbf{e}_{k_2}$. Let $\tilde{\mathbf{z}} = \left[\begin{smallmatrix}\mathbf{0} \\ L_{c_3}\mathbf{l}_6\end{smallmatrix}\right] + \mathbf{z}$, $\gamma_{c_3} = \mathbf{l}_6^T\mathbf{l}_6 + \xi_2$. Let $X_{c_3} = L_{c_3}L_{c_3}^T$,

$$\hat{Z}_{c_3} = Z_{c_1}\begin{bmatrix}I_{k_2-1} & \\ & Q_2^T\end{bmatrix}\begin{bmatrix}\Gamma_6^T & \\ & 1\end{bmatrix}, \tag{2.23}$$

$Y_{c_3} = \left[\begin{smallmatrix}\mathbf{0} & \tilde{\mathbf{z}}(1) \\ Y_c & \hat{Z}_{c_3}(:,1)\end{smallmatrix}\right]$, $Z_{c_3} = \left[\begin{smallmatrix}\tilde{\mathbf{z}}(2:k_2)^T \\ \hat{Z}_{c_3}(:,2:k_2)\end{smallmatrix}\right]$. Moreover,

$$Q_{c_3}^{(k+1)} = Q_{c_1}^{(k+1)}\begin{bmatrix}I_{k_0+k_1+k_2-1} & & \\ & Q_2^T & \\ & & I_{k_1}\end{bmatrix}\begin{bmatrix}I_{k_0+k_1} & & \\ & \Gamma_6^T & \\ & & I_{k_1+1}\end{bmatrix}. \tag{2.24}$$

Then

$$M_{k+1} = Q_{c_3}^{(k+1)^T}A^{(k+1)}Q_{c_3}^{(k+1)} = \begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y_{c_3}^T \\ \mathbf{0} & \mathbf{0} & X_{c_3} & Z_{c_3}^T \\ \mathbf{0} & Y_{c_3} & Z_{c_3} & W_{c_3}\end{bmatrix}\begin{matrix}\}k_0 \\ \}k_1+1 \\ \}k_2-1 \\ \}k_1+1\end{matrix}$$

with

$$W_{c_3} = \begin{bmatrix}\gamma_{c_3} & \hat{Z}_{c_3}(:,k_2+1)^T \\ \hat{Z}_{c_3}(:,k_2+1) & W_c\end{bmatrix}.$$

Remark 2. The proposed algorithm is backward stable because it relies only on Householder and Givens transformations [8].

Remark 3. We observe that the signs of the elements of the antidiagonal of Y and in the diagonal of L can be arbitrarily chosen.

Remark 4. The coefficients c and s of the Givens rotations involved in (2.13) and (2.20), transforming T into an antitriangular matrix, can be computed solving the following equation:

$$c^2\beta^2 - 2sc\alpha + s^2\hat{\gamma} = 0.$$

The parameters c and s can be computed in a stable way from the largest root in absolute value of the quadratic equation [14]

$$\hat{\gamma}t^2 - 2\alpha t + \beta^2 = 0 \quad \text{with } t = s/c.$$

2.2. Computational cost. Given $A^{(k)} = Q^{(k)}M^{(k)}Q^{(k)^T}$, we have just described an algorithm to reduce $A^{(k+1)}$ into proper block antitriangular form by orthogonal transformations. The cost depends on which case happens. Here we examine the cost for each separate case. The only common operation to each case is the product (2.2) requiring $2k^2$ floating point operations.

a. Check whether $\|[\tilde{\mathbf{a}}^T,\ \gamma]\|_2 = 0$ requires $O(k)$ floating point operations.
b. The multiplication (2.3) requires $4k_0k$ floating point operations.

c. The products (2.4), (2.5), (2.6), (2.7), (2.10), (2.11), and (2.12) are common to this case, and their costs are $3k_1^2$, $6k_1k_2$, $6k_1^2$ (due to the symmetry), $6k_1k$, $6k_2^2$, $6k_2k$, and $6k_1k_2$ floating point operations, respectively.
 c.1. No additional floating point operations are required here.
 c.2. The products (2.14), (2.16), (2.17), (2.18), and (2.19) require $3k_2^2$, $6k_2k$, $6k_1k_2$, $3k_1^2$, and $6k_1k$ floating point operations, respectively.
 c.3. The products (2.21), $L_{c_3}\mathbf{l}_6$ in (2.22), (2.23), and (2.24) require $3k_2^2$, k_2^2, $6k_1k_2$, and $6k_2k$ floating point operations, respectively.

Hence, the complexity of the algorithm strongly depends on the inertia of the principal submatrices of the symmetric matrix to be factorized, and it is of order n^3.

The number of operations of the multiplications by the Givens rotations in the proposed algorithm can be halved if the latter rotations are replaced by the fast Givens transformations [2].

2.3. Bounds and inequalities. Let A be an arbitrary symmetric matrix; then we have an antitriangular decomposition of the type

$$Q^TAQ = M = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y^T \\ \mathbf{0} & \mathbf{0} & X & Z^T \\ \mathbf{0} & Y & Z & W \end{bmatrix} \begin{matrix} \}n-(2n_1+n_2) \\ \}n_1 \\ \}n_2 \\ \}n_1 \end{matrix} \qquad ,$$

where $Q \in \mathbb{R}^{n\times n}$ is orthogonal, $W \in \mathbb{R}^{n_1\times n_1}$, $X = \varepsilon LL^T \in \mathbb{R}^{n_2\times n_2}$, and $\varepsilon = \pm 1$ and where the inertia is given by $\{n_1+n_2, n-(2n_1+n_2), n_1\}$ if $\varepsilon = -1$ and by $\{n_1, n-(2n_1+n_2), n_1+n_2\}$ if $\varepsilon = 1$. We want to relate the singular values of Y and L to the eigenvalues of A. We first note that the zero eigenvalues (and corresponding eigenvectors) are correctly identified by this decomposition. For the sake of simplicity, we will therefore exclude them from our discussion, and we will thus assume that we have a nonsingular matrix A with the following decomposition:

$$Q^TAQ = M = \begin{bmatrix} \mathbf{0} & \mathbf{0} & Y^T \\ \mathbf{0} & X & Z^T \\ Y & Z & W \end{bmatrix} \begin{matrix} \}n_1 \\ \}n_2 \\ \}n_1 \end{matrix}$$

with n_1+n_2 eigenvalues of sign ε and n_1 eigenvalues of sign $-\varepsilon$ and with $n = 2n_1+n_2$. We will order the eigenvalues of A (or equivalently of M) and the singular values of L and Y in decreasing order and denote them as follows (here we assume $\varepsilon = 1$):

$$\Lambda(A) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n_1+n_2} > 0 > \lambda_{n_1+n_2+1} \geq \cdots \geq \lambda_n, \tag{2.25}$$

$$\Sigma(L) : \sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n_2} > 0, \tag{2.26}$$

$$\Sigma(Y) : \gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_{n_1} > 0. \tag{2.27}$$

Notice that for the case $\varepsilon = -1$ it suffices to consider $-A$ instead of A, and essentially the same discussion will apply. We will therefore restrict ourselves here to the case $\varepsilon = 1$. The first sets of inequalities are easy to obtain.

THEOREM 2.2. *The eigenvalues of X are given by $\sigma_1^2 \geq \cdots \geq \sigma_{n_2}^2 > 0$ and satisfy the inequalities*

$$0 < \lambda_{i+n_1} \leq \sigma_i^2 \leq \lambda_i, \quad i = 1, \ldots, n_2.$$

Proof. To prove it, just apply the Cauchy interlacing theorem to the $(n_1+n_2)\times(n_1+n_2)$ principal submatrix

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & X \end{bmatrix}$$

of M, whose largest n_2 eigenvalues are the $\sigma_i^2,\ i=1,\dots,n_2$. ∎

We now look at another submatrix of M.

THEOREM 2.3. *The ordered eigenvalues*

$$\hat\lambda_1 \geq \cdots \geq \hat\lambda_{n_1} > 0 > \hat\lambda_{n_1+1} \geq \cdots \geq \hat\lambda_{2n_1}$$

of the $2n_1\times 2n_1$ *submatrix* $\left[\begin{smallmatrix} \mathbf{0} & Y^T \\ Y & W \end{smallmatrix}\right]$ *of* M *satisfy the following inequalities:*

$$0 < \lambda_{i+n_2} \leq \hat\lambda_i \leq \lambda_i, \quad i=1,\dots,n_1, \tag{2.28}$$

$$\lambda_{i+n_1+n_2} \leq \hat\lambda_{n_1+i} < 0, \quad i=1,\dots,n_1. \tag{2.29}$$

Proof. All the inequalities follow from applying the Cauchy interlacing inequalities to the given submatrix of M. But the upper bounds of the inequalities (2.29) are replaced by zero because this is also guaranteed by our decomposition. ∎

We now relate the $2n_1$ eigenvalues $\hat\lambda_i$ to the n_1 singular values γ_i of the matrix Y.

THEOREM 2.4. *The ordered singular values*

$$\gamma_1 \geq \cdots \geq \gamma_{n_1} > 0$$

of the $n_1\times n_1$ *submatrix* Y *of* M *satisfy the following inequalities and equality:*

$$\prod_{i=1}^{j}\gamma_i^2 \leq \prod_{i=1}^{j}\hat\lambda_i \prod_{i=1}^{j}|\hat\lambda_{2n_1-i+1}|, \quad j=1,\dots,n_1-1, \tag{2.30}$$

$$\prod_{i=1}^{n_1}\gamma_i^2 = \prod_{i=1}^{n_1}\hat\lambda_i \prod_{i=1}^{n_1}|\hat\lambda_{2n_1-i+1}|. \tag{2.31}$$

Proof. Without loss of generality, we can assume that Y is diagonal and ordered

$$Y = \mathrm{diag}\{\gamma_1,\dots,\gamma_{n_1}\}.$$

Consider the $2j\times 2j$ submatrix (made of the leading $j\times j$ subblocks of Y and W):

$$\begin{bmatrix} \mathbf{0} & Y_j^T \\ Y_j & W_j \end{bmatrix} \quad \text{of} \quad \begin{bmatrix} \mathbf{0} & Y^T \\ Y & W \end{bmatrix}.$$

Its eigenvalues $\tilde\lambda_1 \geq \cdots \geq \tilde\lambda_j > 0 > \tilde\lambda_{j+1} \geq \cdots \geq \tilde\lambda_{2j}$ satisfy the bounds

$$0 < \tilde\lambda_i \leq \hat\lambda_i, \quad i=1\dots j, \quad \text{and} \quad \hat\lambda_{2n_1+1-i} \leq \tilde\lambda_{2j+1-i} < 0, \quad i=1\dots j. \tag{2.32}$$

But the product $\prod_{i=1}^{j}\tilde\lambda_i\prod_{i=1}^{j}|\tilde\lambda_{2j-i+1}|$ of the absolute values of these $2j$ eigenvalues is also to equal to $\prod_{i=1}^{j}\gamma_i^2$ since this is the absolute value of the determinant of the corresponding matrix. Combining this with the Cauchy interlacing inequalities (2.32) yields the desired inequalities (2.30). The product equality (2.31) follows from the expression of the determinant of the whole matrix. ∎

We now use this to obtain bounds involving the eigenvalues of A.

THEOREM 2.5. *The ordered singular values*

$$\gamma_1 \geq \cdots \geq \gamma_{n_1} > 0$$

of the $n_1 \times n_1$ submatrix Y of M satisfy the following inequalities:

$$\prod_{i=1}^{j} \gamma_i^2 \leq \prod_{i=1}^{j} \lambda_i \prod_{i=1}^{j} |\lambda_{2n_1+n_2-i+1}|, \quad j = 1, \ldots, n_1. \tag{2.33}$$

Proof. This follows trivially from combining Theorems 2.3 and 2.4. ◻

Let us suppose that the matrix M is sparse with the matrices

$$L = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \sigma_{n_2} \end{bmatrix}, \; W = \begin{bmatrix} \omega_1 & 0 & \ldots & 0 \\ 0 & \omega_2 & \ddots & 0 \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & \omega_{n_1} \end{bmatrix},$$

$$Y = \begin{bmatrix} 0 & \ldots & 0 & \gamma_1 \\ \vdots & \iddots & \gamma_2 & 0 \\ 0 & \iddots & \iddots & \vdots \\ \gamma_{n_1} & 0 & \ldots & 0 \end{bmatrix}$$

as the only nonzero blocks. The eigenvalues of such a matrix M are easily seen to be $\sigma_i^2, i = 1, \ldots, n_2$, and the positive and negative eigenvalues of the 2×2 matrices $\left[\begin{smallmatrix} 0 & \gamma_i \\ \gamma_i & \omega_i \end{smallmatrix}\right], i = 1, \ldots, n_1$, i.e., $\hat{\lambda}_i,\ i = 1, \ldots, 2n_1$. If we choose the ordering

$$\lambda_i = \begin{cases} \hat{\lambda}_i, & i = 1, \ldots, n_1, \\ \sigma_{i-n_1}^2 & i = n_1 + 1, \ldots, n_1 + n_2, \\ \hat{\lambda}_{i-n_2}, & i = n_1 + n_2 + 1, \ldots, 2n_1 + n_2, \end{cases}$$

then the upper bounds of Theorem 2.5 become equalities. Notice that the λ_i may not be ordered in a decreasing way. It also turns out that the matrices M obtained by our reduction algorithm are in fact reasonably near to such a sparse matrix M. This means that the above diagonal approximation in general gives good estimates of the true eigenvalues of A. This will be illustrated in section 4 with numerical experiments.

We now prove that if A has only one (repeated) negative eigenvalue λ_- and one (repeated) positive eigenvalue λ_-, then the antitriangular form must have the above sparse structure, where, moreover, the σ_i, γ_i, and ω_i are constant, apart from the signs of γ_i, which can change.

LEMMA 2.6. *If A has only one (repeated) negative eigenvalue λ_- and one (repeated) positive eigenvalue λ_+, then the proper block antitriangular form is unique up to a sign matrix*

$$S = \begin{bmatrix} \pm 1 & & \\ & \ddots & \\ & & \pm 1 \end{bmatrix} \in \mathbb{R}^{n_1 \times n_1}$$

and given by

$$Q^T A Q = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \gamma S J_{n_1} \\ \mathbf{0} & \lambda_+ I_{n_2} & \mathbf{0} \\ \gamma J_{n_1} S & \mathbf{0} & \omega I_{n_1} \end{bmatrix},$$

where $\gamma = \sqrt{-\lambda_- \lambda_+}$, $\omega = \lambda_- + \lambda_+$, J_{n_1} is the antiidentity of order n_1.

Proof. Without loss of generality, we suppose that the algebraic multiplicities of λ_- and λ_+ are n_1 and $n_1 + n_2$, respectively, with $2n_1 + n_2 = n$. Let

$$A = V \begin{bmatrix} \lambda_+ I_{n_1+n_2} & \\ & \lambda_- I_{n_1} \end{bmatrix} V^T$$

be the eigenvalue decomposition of A with $V \in \mathbb{R}^{n\times n}$ orthogonal. For simplicity, we consider the coordinate system where $V = I_n$, i.e.,

$$A = \begin{bmatrix} \lambda_+ I_{n_1} & & \\ & \lambda_+ I_{n_2} & \\ & & \lambda_- I_{n_1} \end{bmatrix}.$$

Any neutral subspace $\mathcal{U}_n$ of maximal dimension then has dimension n_1 and is formed by the vectors $\mathbf{x}$ such that $\mathbf{x}^T A\mathbf{x} = 0$. If we normalize these vectors such that $\|\mathbf{x}\|_2 = 1$, then

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \\ \mathbf{x}_3 \end{bmatrix} \begin{matrix} \}n_1 \\ \}n_2 \\ \}n_1 \end{matrix}, \quad \left\| \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \right\|_2 = c, \ \|\mathbf{x}_3\|_2 = s,$$

where $c = \sqrt{-\lambda_-}/\sqrt{\lambda_+ - \lambda_-}$, $s = \sqrt{\lambda_+}/\sqrt{\lambda_+ - \lambda_-}$. A basis for this space must therefore be given by

$$Q_n = \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} \begin{bmatrix} cI_{n_1} \\ \mathbf{0}_{n_2} \\ -sI_{n_1} \end{bmatrix}$$

with $Q_1 \in \mathbb{R}^{(n_1+n_2)\times(n_1+n_2)}$ and $Q_2 \in \mathbb{R}^{n_1\times n_1}$ orthogonal matrices. Any nonnegative subspace $\mathcal{U}_+$ of maximal dimension containing $\mathcal{U}_n$ then has dimension $n_1 + n_2$ and must have an orthogonal basis

$$Q_+ = \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} \left[\begin{array}{r|c} cI_{n_1} & s\hat{X} \\ \mathbf{0}_{n_2} & \hat{Y} \\ -sI_{n_1} & c\hat{X} \end{array}\right]$$

with $\hat{X}^T\hat{X} + \hat{Y}^T\hat{Y} = I_{n_2}$. But

$$\begin{aligned} Q_+^T A Q_+ &= \left[\begin{array}{r|c} cI_{n_1} & s\hat{X} \\ \mathbf{0}_{n_2} & \hat{Y} \\ -sI_{n_1} & c\hat{X} \end{array}\right]^T \begin{bmatrix} \lambda_+ I_{n_1} & & \\ & \lambda_+ I_{n_2} & \\ & & \lambda_- I_{n_1} \end{bmatrix} \left[\begin{array}{r|c} cI_{n_1} & s\hat{X} \\ \mathbf{0}_{n_2} & \hat{Y} \\ -sI_{n_1} & c\hat{X} \end{array}\right] \\ &= \begin{bmatrix} \mathbf{0} & cs(\lambda_+ - \lambda_-)\hat{X} \\ cs(\lambda_+ - \lambda_-)\hat{X}^T & \lambda_+ \hat{Y}^T\hat{Y} + (\lambda_+ s^2 + \lambda_- c^2)\hat{X}^T\hat{X} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \sqrt{-\lambda_+\lambda_-}\hat{X} \\ \sqrt{-\lambda_+\lambda_-}\hat{X}^T & \lambda_+ I + \lambda_- \hat{X}^T\hat{X} \end{bmatrix}. \end{aligned}$$

It is easily seen that this is positive semidefinite only if $\hat{X} = 0$, and hence $\hat{Y}$ is a $n_2 \times n_2$ orthogonal matrix Q_3. Therefore, we have

$$Q_+ = \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} \begin{bmatrix} I_{n_1} & & \\ & Q_3 & \\ & & I_{n_1} \end{bmatrix} \left[\begin{array}{r|c} cI_{n_1} & \mathbf{0} \\ \mathbf{0}_{n_2} & I_{n_2} \\ -sI_{n_1} & \mathbf{0} \end{array} \right].$$

Finally, any orthogonal completion must then be of the form

$$Q = \begin{bmatrix} \hat{Q}_1 & \\ & Q_2 \end{bmatrix} \begin{bmatrix} cI_{n_1} & \mathbf{0} & sI_{n_1} \\ \mathbf{0} & I_{n_2} & \mathbf{0} \\ -sI_{n_1} & \mathbf{0} & cI_{n_1} \end{bmatrix} \begin{bmatrix} I_{n_1+n_2} & \\ & SJ_{n_1} \end{bmatrix}, \; \hat{Q}_1 = Q_1 \begin{bmatrix} I_{n_1} & \\ & Q_3 \end{bmatrix},$$

so that after applying this general transformation, we find indeed the required antitriangular shape

$$Q^T AQ = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \gamma SJ_{n_1} \\ \mathbf{0} & \lambda_+ I_{n_2} & \mathbf{0} \\ \gamma SJ_{n_1} & \mathbf{0} & \omega I_{n_1} \end{bmatrix}. \qquad \square$$

Remark 5. Notice that Q is not unique, but the resulting proper block antitriangular matrix is, up to the sign matrix S.

3. Rank-one modification. In many applications it is important to update (downdate) a symmetric indefinite matrix $A \in \mathbb{R}^{n\times n}$, whose dominant/minor subspace is easily detectable, with a symmetric rank-one modification in a fast way. Algorithms are available to reduce rank-one modifications of symmetric tridiagonal matrices [22, 23], of diagonal ones [4, 7, 11], and of symmetric diagonal plus semiseparable ones [18, 19] in tridiagonal form. Let the symmetric matrix A of order n be factorized as $A = QTQ^T$, where T is one of the latter matrices and Q is an orthogonal one. Although the rank-one modification of the matrix T can be computed in $O(n^2)$ flops, the updating of the Q factor requires $O(n^3)$ flops.

The updating/downdating of the symmetric *rank-revealing* decomposition $A = QR^T \Omega RQ^T$ with Q orthogonal, R upper triangular, and $\Omega = \mathrm{diag}(\pm 1)$ modified by a symmetric rank-one matrix, described in [13], requires $O(n^2)$ flops. However, it involves hyperbolic transformations. These transformations can introduce severe cancellation in the updated matrix even when choosing careful implementations [3].

In this section we show that a symmetric proper block antitriangular matrix can be updated/downdated with a symmetric rank-one modification with $O(n^2)$ flops in a stable way.

Let $\mathbf{y} \in \mathbb{R}^n$ and let $A = QMQ^T$ with $\mathrm{Inertia}(A) = (n_-, n_0, n_+)$, $n_- + n_0 + n_+ = n$, and Q orthogonal. Let $n_1 = \min(n_-, n_+)$ and $n_2 = \max(n_-, n_+) - n_1$,

$$M = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & Y^T \\ \mathbf{0} & \mathbf{0} & X & Z^T \\ \mathbf{0} & Y & Z & W \end{bmatrix} \begin{matrix} \}n_0 \\ \}n_1 \\ \}n_2 \\ \}n_1 \end{matrix}$$

with $Y \in \mathbb{R}^{n_1 \times n_1}$ nonsingular lower antitriangular and $X \in \mathbb{R}^{n_2 \times n_2}$, $X = \varepsilon LL^T$ with $\varepsilon = 1$ if $n_+ > n_-$, $\varepsilon = -1$ if $n_+ < n_-$, and L lower triangular. The aim is to

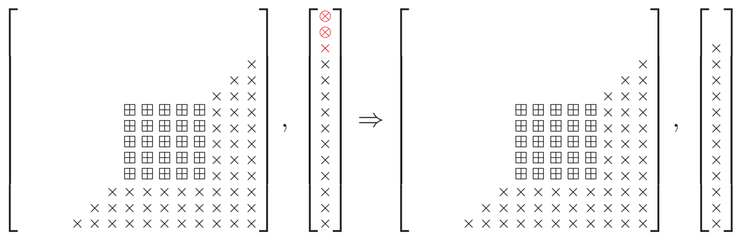

FIG. 3.1. *First step of the algorithm. Application of the Householder similarity transformation annihilating the first $n_0 - 1$ entries in the vector $\mathbf{x}$. For the sake of simplicity, only the block antitriangular matrix M and the vector $\hat{\mathbf{x}}$ are displayed. The entries to be annihilated are denoted by $\otimes$, and the entries modified by the multiplication are in red. The entries of the central definite submatrix X are denoted by the symbol $\boxplus$.*

transform $QMQ^T \pm \mathbf{y}\mathbf{y}^T = Q(M \pm Q^T\mathbf{y}\mathbf{y}^TQ)Q^T$, with M symmetric proper block antitriangular, into $\hat{Q}\hat{M}\hat{Q}^T$, a matrix with the same structure of M and $\hat{Q}$ orthogonal, annihilating the entries of the vector $\mathbf{x} = Q^T\mathbf{y}$.

Let $N_1 = n_0 + n_1$ and $N_2 = n_0 + n_1 + n_2$. We first consider the case $n_0, n_1, n_2 > 0$. The case $n_0 = 0$ is shortly described after.

We divide the algorithm into four different steps.

3.1. First step. If $n_0 > 1$, the first step is to determine a Householder matrix $H^{(1)}$ such that the product $H^{(1)}\mathbf{x}(1 : n_0)$ has annihilated all the entries but the last one. Let $H_1 = \mathrm{diag}(H^{(1)}, I_{n-n_0})$. Then

$$Q(M \pm \mathbf{x}\mathbf{x}^T)Q^T = QH_1^T(H_1MH_1^T \pm H_1\mathbf{x}\mathbf{x}^TH_1^T)H_1Q^T$$
$$= Q_1(M \pm \hat{\mathbf{x}}\hat{\mathbf{x}}^T)Q_1^T$$

with $Q_1 = QH_1^T$, $\hat{\mathbf{x}} = H_1\mathbf{x}$. We observe that M remains unchanged since its first n_0 rows/columns are zero. This reduction is depicted in Figure 3.1 for a proper block antitriangular matrix with inertia $(3, 3, 8)$. We remark that the algorithm reverts to the $n_0 = 0$ case if $\|\mathbf{x}(1 : n_0)\|_2 = 0$.

Computational complexity. Due to the special structure of the Householder matrix H_1, the multiplication $Q_1 = QH_1^T$ requires $4n_0n$ flops and $\hat{\mathbf{x}} = H_1\mathbf{x}$ requires $4n_0$. The number of operations could be reduced to $3n_0n$ and $3n_0$ by using a sequence of fast Givens transformations [2] rather than a single Householder transformation.

3.2. Second step. If $n_1 = \min(n_-, n_+) > 0$, i.e., if M is indefinite, in this step the entries $n_0, n_0 + 1, \ldots, N_1 - 1$ of $\hat{\mathbf{x}}$ are annihilated by means of multiplication by a sequence of n_1 Givens rotations G_i, $i = 1, 2, \ldots, n_1$, such that G_i acts on the $(n_0 + i - 1)$th and $(n_0 + i)$th rows of $G_{i-1} \cdots G_1\hat{\mathbf{x}}$, annihilating the entry in position $(n_0 + i - 1)$. Let $M^{(0)} = M$, and $M^{(i)} = G_iM^{(i-1)}G_i^T$, $\hat{\mathbf{x}} = G_i\hat{\mathbf{x}}$, $i = 1, \ldots, n_1$. The product $G_iM^{(i-1)}G_i^T$ modifies the last i entries of the rows/columns $n_0 + i - 1$ and $n_0 + i$ of $M^{(i-1)}$ introducing a nonzero entry in position $(n_0 + i - 1, n - i + 1)$ and, symmetrically, in position $(n - i + 1, n_0 + i - 1)$.

The second step is depicted in Figure 3.2. For the sake of simplicity, the first $n_0 - 1$ entries of x and the first $n_0 - 1$ rows and columns of M are not depicted since they are zero.

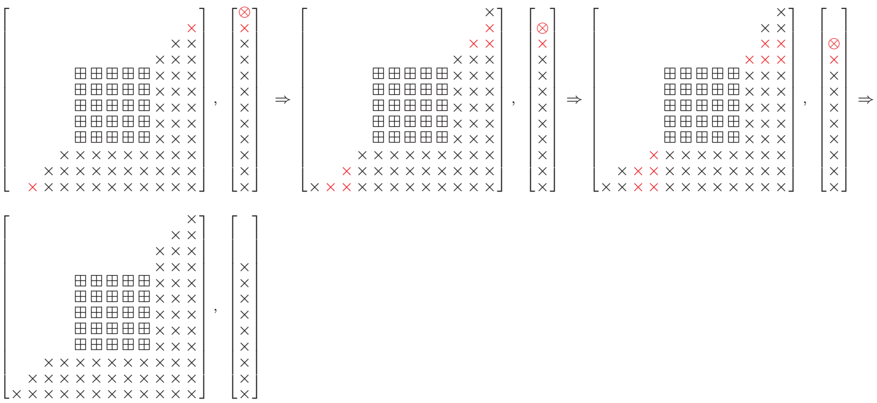

FIG. 3.2. *Second step of the algorithm. Application of a sequence Givens rotations annihilating the entries $n_0, n_0+1, \ldots, N_1-1$ of the vector $\hat{\mathbf{x}}$.*

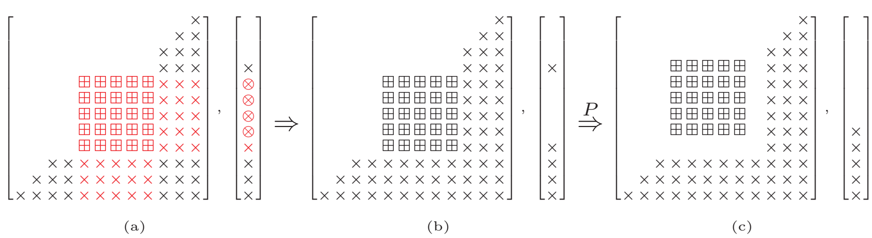

FIG. 3.3. *Third step of the algorithm. Application of the sequence of Givens rotations annihilating all the entries but the last one of $\hat{\mathbf{x}}$ corresponding to the central definite submatrix X.*

Computational complexity. Due to the antitriangular structure of $M^{(0)}$, the computation of $M^{(n_1)} = G_{n_1} G_{n_1-1} \cdots G_1 M^{(0)} G_1^T \cdots G_{n_1-1}^T G_{n_1}^T$, requires $3n_1^2$ flops. Moreover, the computation of $Q_2 = Q_1 G_1^T \cdots G_{n_1-1}^T G_{n_1}^T$ requires $6n_1 n$ flops.

3.3. Third step. The aim of this step is to annihilate all the entries but the last one of $\hat{\mathbf{x}}$ corresponding to the central definite submatrix X by a sequence of Givens rotations. This step is depicted in Figure 3.3 ((a)⇒(b)). Since X is in factored form, i.e., $X = \varepsilon L L^T$, with $\varepsilon = \pm 1$, another sequence of *inner* Givens rotations is considered to preserve the factored structure. In particular, the first Givens rotation $\check{G}_1$ is applied to $\hat{\mathbf{x}}$, modifying the rows N_1+1 and N_1+2 and annihilating the entry N_1+1. Then the similarity transformation is computed $M^{(n_1+1)} = \check{G}_1 M^{(n_1)} \check{G}_1^T$. We observe that this transformation modifies the first two rows/columns of the submatrix X creating a nonzero entry in the position $(1,2)$ of L. To remove this bulge a new Givens rotation $\breve{G}_1$ is applied to the right of L. We observe that the latter rotation acts only on the matrix L. The described procedure is depicted in Figure 3.4. Without loss of generality only the central submatrix X and the corresponding entries of the vector $\hat{\mathbf{x}}$ are depicted. Let $M^{(n_1+n_2-1)} = \check{G}_{n_2-1} \cdots \check{G}_1 M^{(n_1)} \check{G}_1^T \cdots \check{G}_{n_2-1}^T$.

Finally, the row (column) N_1 of $M^{(n_1+n_2-1)}$ is moved below (to the right of) the definite submatrix $M^{(n_1+n_2-1)}(N_1+1:N_2, N_1+1:N_2)$ by the multiplication of a permutation matrix P. Let $M^{(n_1+n_2)} = P M^{(n_1+n_2-1)} P^T$ and $\hat{\mathbf{x}} = P\hat{\mathbf{x}}$. This transformation is depicted in Figure 3.3 ((b) ⇒ (c)).

Computational complexity. Due to the antitriangular structure, the computation of $M^{(n_1+n_2-1)}$ requires $6n_1n_2 + 3n_2^2$ flops. Moreover, to remove the bulges of L in

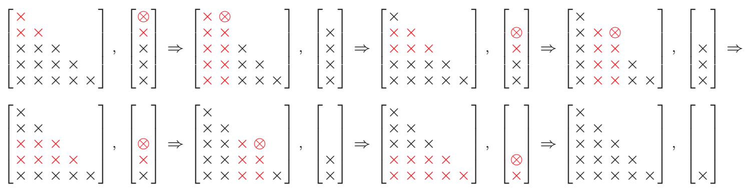

FIG. 3.4. *Third step of the algorithm. To preserve the Cholesky structure, each multiplication by an outer Givens rotation $\breve{G}_i$, introducing a bulge in the lower triangular structure of L, is followed by a multiplication by an inner Givens rotation $\check{G}_i$ removing the bulge.*

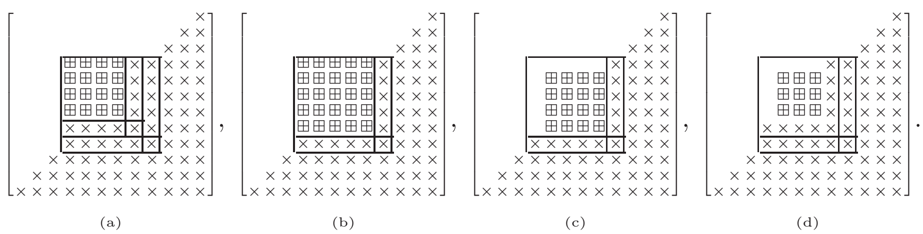

FIG. 3.5. *Fourth step. Reduction of the matrix $\hat{M}$* (a) *to the matrix either* (b), (c), *or* (d) *if the submatrix $\hat{M}(N_1 : N_2 - 1, N_1 : N_2 - 1)$ is definite, singular, or indefinite.*

order to restore the lower triangular structure requires $3n_2^2$ flops. Furthermore, the computation of $Q_4 = Q_3 \check{G}_1^T \cdots \check{G}_{n_2-1}^T$ requires $6n_2 n$ flops.

3.4. Fourth step. The rank-one matrix $\hat{\mathbf{x}}\hat{\mathbf{x}}^T$ is now added or subtracted to $M^{(n_1+n_2)}$,

$$\hat{M} = M^{(n_1+n_2)} \pm \hat{\mathbf{x}}\hat{\mathbf{x}}^T.$$

Since $\hat{\mathbf{x}}(1 : N_2 - 2) = \mathbf{0}$, $\hat{M}$ differs from $M^{(n_1+n_2)}$ for the submatrix $\hat{M}(N_2 - 1 : N_2 + n_1, N_2 - 1 : N_2 + n_1)$. Although $\hat{M}(N_1 : N_2 - 2, N_1 : N_2 - 2)$ is still definite, nothing can be said about the definiteness of the submatrix $\hat{M}(N_1 : N_2, N_1 : N_2)$.

To complete the reduction, it is sufficient to reduce to proper block antitriangular form the submatrix $\hat{M}(N_1 : N_2, N_1 : N_2)$, by applying either one step of the algorithm described in section 2 starting from the symmetric definite submatrix $\hat{M}(N_1 : N_2 - 2, N_1 : N_2 - 2)$ if the submatrix $\hat{M}(N_1 : N_2 - 1, N_1 : N_2 - 1)$ is singular or two steps otherwise. Of course, the Givens rotations must be applied to the whole matrix $\hat{M}$.

Let $\hat{M}^{(1)}$ be the matrix obtained after one step of the reduction.

For the sake of clarity, we describe the first of the two steps distinguishing the three different cases and the second step in only one subcase.

Case a. $\hat{M}(N_1 : N_2 - 1, N_1 : N_2 - 1)$ definite. Applying the procedure c.1 of section 2.1, the latter matrix is transformed to proper block antitriangular form. This is graphically depicted in Figure 3.5 ((a) ⇒(b)).

Case b. $\hat{M}(N_1 : N_2 - 1, N_1 : N_2 - 1)$ singular. Applying the procedure c.2 of section 2.1, the latter matrix is transformed to proper block antitriangular form. This is graphically depicted in Figure 3.5 ((a) ⇒(c)).

Case c. $\hat{M}(N_1 : N_2 - 1, N_1 : N_2 - 1)$ indefinite. Applying the procedure c.3 of section 2.1, the latter matrix is transformed to proper block antitriangular form. This is graphically depicted in Figure 3.5 ((a) ⇒(d)). To complete this case and the whole reduction, a similarity transformation based on a Givens rotation acting on the rows

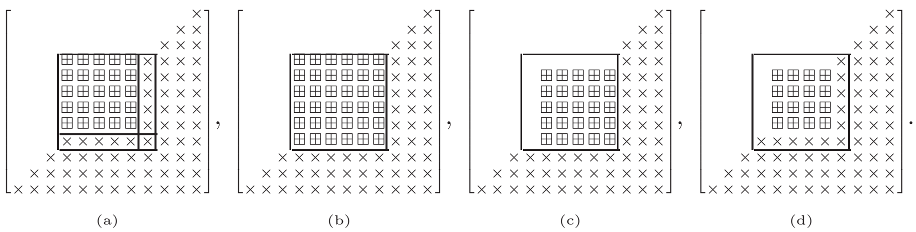

FIG. 3.6. *Fourth step. Reduction of the matrix* $\hat{M}^{(1)}$ (a) *to the matrix either* (b), (c), *or* (d) *if the submatrix* $\hat{M}^{(1)}(N_1 : N_2 - 1, N_1 : N_2 - 1)$ *is definite, singular, or indefinite.*

(columns) $N_2 - 1$ and N_2 in order to annihilate the entry $(N_2 - 1, N_1)$ $((N_1, N_2 - 1))$ is considered.

In what follows we describe the second step of the reduction of $\hat{M}^{(1)}$ to proper block antitriangular form for Case a ($\hat{M}^{(1)}(N_1 : N_2 - 1, N_1 : N_2 - 1)$ definite). Case c ($\hat{M}^{(1)}(N_1 : N_2 - 1, N_1 : N_2 - 1)$ indefinite) can be handled in a similar way.

We distinguish the following three subcases:

1. Subcase a.1. $\hat{M}^{(1)}(N_1 : N_2, N_1 : N_2)$ definite. Applying the procedure c.1 of section 2.1, the latter matrix is transformed to proper block antitriangular form. This is graphically depicted in Figure 3.6 ((a) $\Rightarrow$(b)).
2. Subcase a.2. $\hat{M}^{(1)}(N_1 : N_2, N_1 : N_2)$ singular. Then, the matrix $\hat{M}^{(1)}$ is singular. Applying the procedure c.3 of section 2.1, the latter matrix is transformed to proper block antitriangular form. This is graphically depicted in Figure 3.6 ((a) $\Rightarrow$(c)). Then, as described in c.3 of section 2.1, a sequence of n_1 Givens rotations $\bar{G}_i,\ i = 1, \dots, n_1$, each of those acting on the rows (columns) $N_1 - i$ and $N_1 - i + 1$ annihilating the entry $(N_1 - i, n_1 - i + 1)$, must be applied in order to reduce the whole matrix in proper block antitriangular form.
3. Subcase a.3. $\hat{M}(N_1 : N_2, N_1 : N_2)$ indefinite. Applying the procedure c.3 of section 2.1, the latter matrix is transformed to proper block antitriangular form. This is graphically depicted in Figure 3.6 ((a) $\Rightarrow$(d)).

Computational complexity. The computational complexity of the initial substep is mainly due to the reduction to proper block antitriangular form of the central part of $\hat{M}$ requiring, according to the section 2.1, case c, $O(n_2 n_1 + n^2)$ flops. Moreover, to update the Q matrix, $O(n_2 n)$ flops are required. Another $O(n_1 n)$ flops must be added in subcase a.2, due to the multiplication of the sequence $\bar{G}_i,\ i = 1, \dots, n_1$, by the orthogonal matrix of the factorization.

3.5. Case $n_0 = 0$. If $n_0 = 0$, step 1 of the above described algorithm is skipped. Furthermore, step 2 is modified as follows. Let $M^{(0)} = M$. A sequence of Givens rotations $\mathcal{G}_i,\ i = 1, \dots, n_1 - 1$, is considered in order to annihilate the first $n_1 - 1$ entries of $\mathbf{x}$. Let $M^{(i)} = \mathcal{G}_i M^{(i-1)} \mathcal{G}_i^T$. The matrix $M^{(n_1-1)}$ differs from a proper block antitriangular matrix for the entries $(i, 2n_1 + n_2 - i)$ and $(2n_1 + n_2 - i, i)$ that are different from zero for $i = 1, \dots, n_1 - 1$. In order to annihilate the latter entries another sequence $\tilde{\mathcal{G}}_i,\ i = 1, \dots, n_1 - 1$, of Givens rotations, each of those acting on the rows and columns $2n_1 + n_2 - i$ and $2n_1 + n_2 - i + 1$, is considered such that

$$M^{(n_1+i-1)} = \tilde{\mathcal{G}}_i M^{(n_1+i-2)} \tilde{\mathcal{G}}_i^T,\ i = 1, \dots, n_1 - 1.$$

This step is graphically depicted in Figure 3.7 ((a)$\Rightarrow$(e)). Then, the reduction proceeds similarly as for the case $n_0 \neq 0$ (see Figure 3.7 (e)$\Rightarrow$(f)$\Rightarrow$(g)).

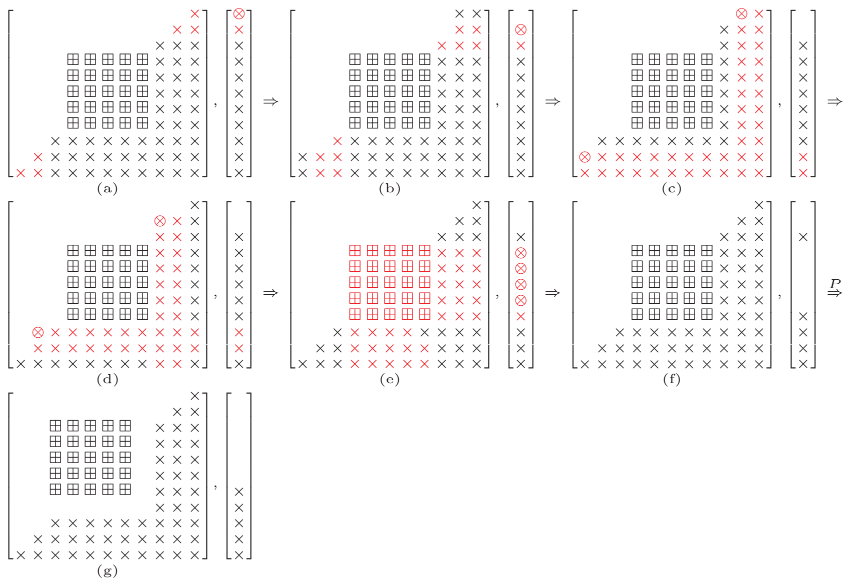

FIG. 3.7. *Case* $n_0 = 0$.

4. Numerical experiments. Some numerical experiments showing the properties of the proposed algorithm are reported in this section. In particular, it is shown that the numerical results agree with the bounds described in section 2.3. The experiments are carried out in MATLAB. In all the examples we denote by A the initial symmetric indefinite matrix, by M the similar block antitriangular matrix computed by the proposed algorithm, by Q the computed orthogonal matrix such that $A = Q^T MQ$, and by M_d the symmetric indefinite matrix obtained by extracting the main diagonal from M and the antidiagonal from Y. Moreover, we denote by $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_{n-1}(A) \geq \lambda_n(A)$ and by $\lambda_1(M_d) \geq \lambda_2(M_d) \geq \cdots \geq \lambda_{n-1}(M_d) \geq \lambda_n(M_d)$ the eigenvalues of A and M_d, respectively. When M_d is close to M, obviously $\sigma_i^2,\ i = 1, \ldots, n_2$, are good approximations of n_2 eigenvalues of M and $|\lambda_i(A)\lambda_{n-i+1}(A)|,\ i = 1, \ldots, n_1$, are good approximations of γ_i^2, with σ_i and γ_i defined in subsection 2.3, for an appropriate ordering of the eigenvalues of A. We chose the tolerance $\tau = 1.0 \times 10^{-15}$ for the binary comparisons in the examples. The considered examples illustrate well some of the properties which we want to highlight. The first example is characterized by having two clusters of eigenvalues. As a consequence (see Lemma 2.6), the eigenvalues of M_d approximate well those of A. The second example shows that if the entries of the matrix are chosen randomly, we have almost the same number of positive and negative eigenvalues. Moreover, the computed proper block antitriangular matrix is almost antidiagonal dominant. The third and fourth example are taken from the Matrix Market [15]. The eigenvalue estimation property for these more practical examples is less pronounced, except for the last example which is diagonal dominant.

Example 1. In this example $n = 100$, $d = [-15 \times \texttt{ones}(40, 1); 25 \times \texttt{ones}(60, 1)] + .5\times\texttt{randn}(n, 1)$ and $A = \tilde{Q}\texttt{diag}(d)\tilde{Q}^T$ with $\tilde{Q}$ a random orthogonal matrix. Therefore, 40 eigenvalues of A are clustered around -15 and 60 around 25. The computed matrix M is depicted in Figure 4.1. The eigenvalues of A and M_d are depicted in Figure 4.2. The backward error is $\|A - QMQ^T\|_2 = 8.68 \times 10^{-14}$.

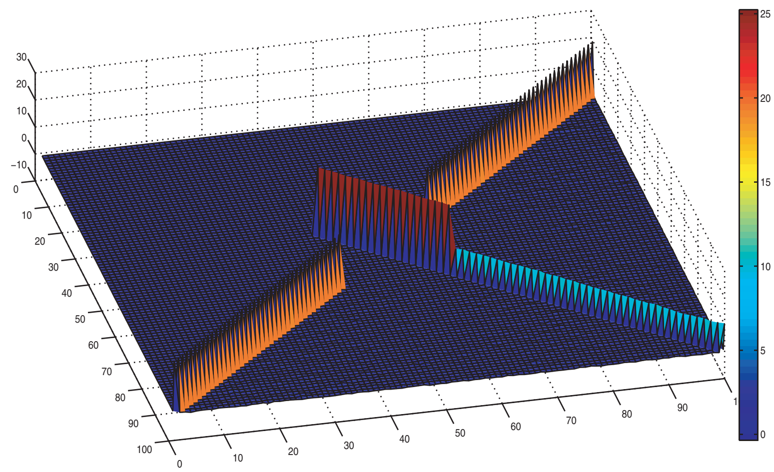

FIG. 4.1. *Plot of the entries of the block antitriangular matrix M of Example 1 computed by the proposed algorithm.*

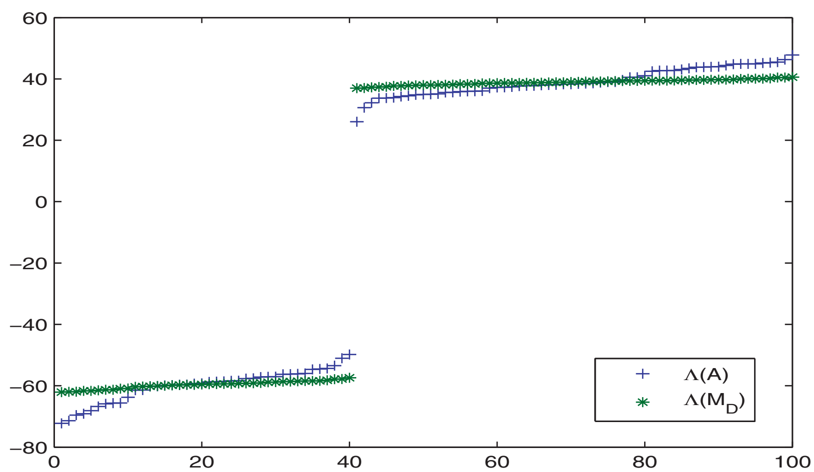

FIG. 4.2. *Eigenvalues of A and M_d of Example 1.*

Example 2. Let $n = 100$, $B = \texttt{randn}(100)$, and $A = B + B^T$. The matrix M computed by the proposed algorithm is depicted in Figure 4.3. The eigenvalues of A and M_d are depicted in Figure 4.4. The backward error is $\|A - QMQ^T\|_2 = 7.42 \times 10^{-14}$.

Example 3. The matrix A considered in this example is the `FIDAPM05` matrix generated by FIDAP and available at the Matrix Market [15] and depicted in Figure 4.5. Its order is $n = 42$, and it has 14 negative eigenvalues, 1 zero, and 27 positive ones. The computed matrix M is displayed in Figure 4.6. The computed inertia is (14,1,27). The eigenvalues of A and M_d are depicted in Figure 4.7. The backward error is $\|A - QMQ^T\|_2 = 1.84 \times 10^{-15}$.

Example 4. The matrix A considered in this example is the real part of the complex symmetric matrix called `QC324`, available at the Matrix Market [15], modeling H_2^+ in an Electromagnetic Field, and depicted in Figure 4.8. Its order is $n = 324$,

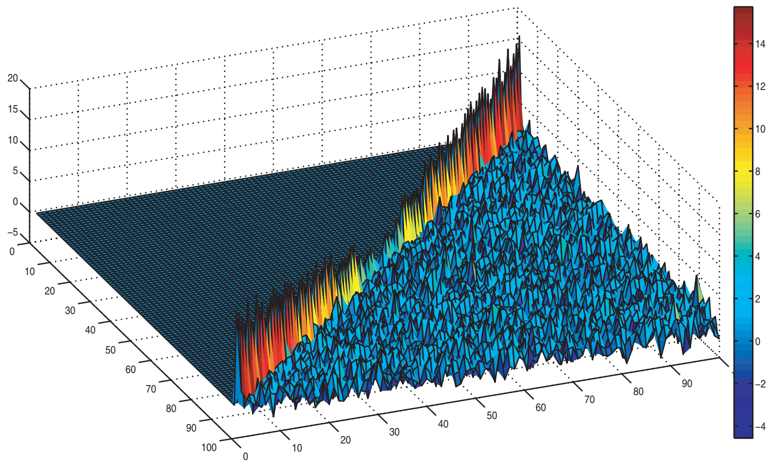

FIG. 4.3. *Plot of the entries of matrix M of Example* 2.

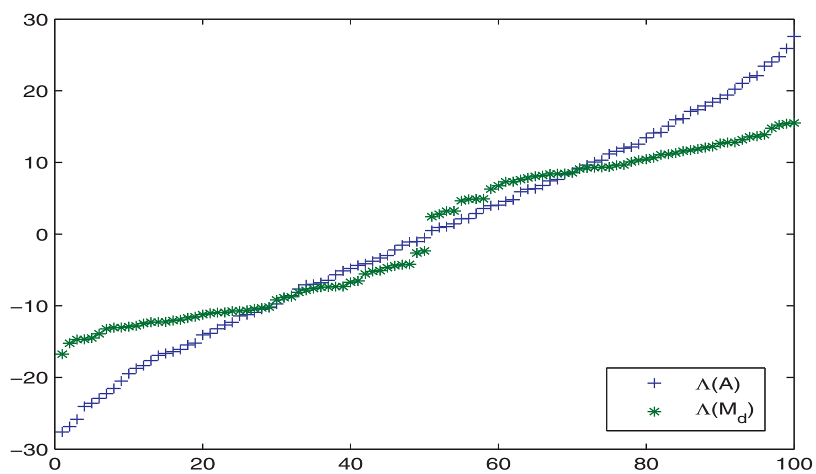

FIG. 4.4. *Eigenvalues of A and M_d of Example* 2.

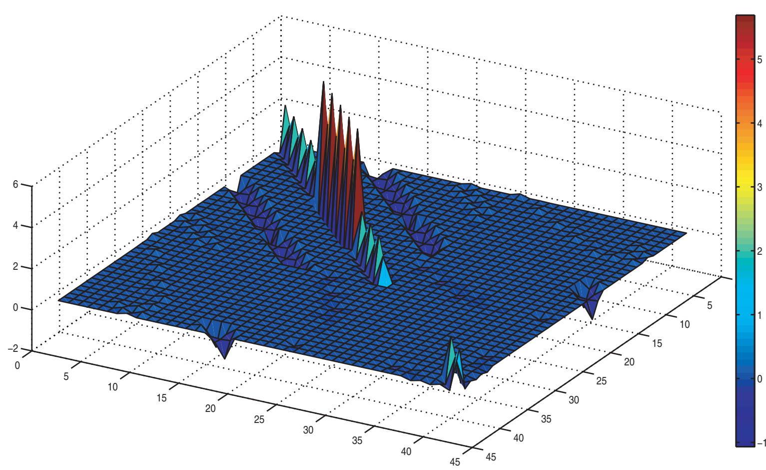

FIG. 4.5. *Plot of the entries of matrix A of Example* 3.

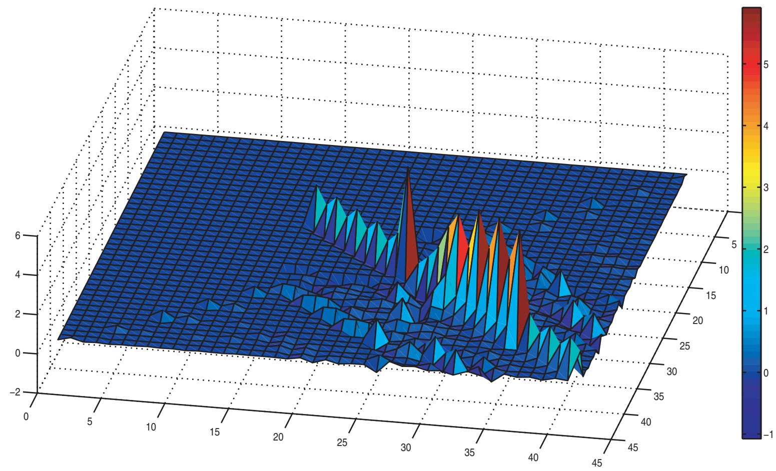

FIG. 4.6. *Plot of the entries of matrix M of Example* 3.

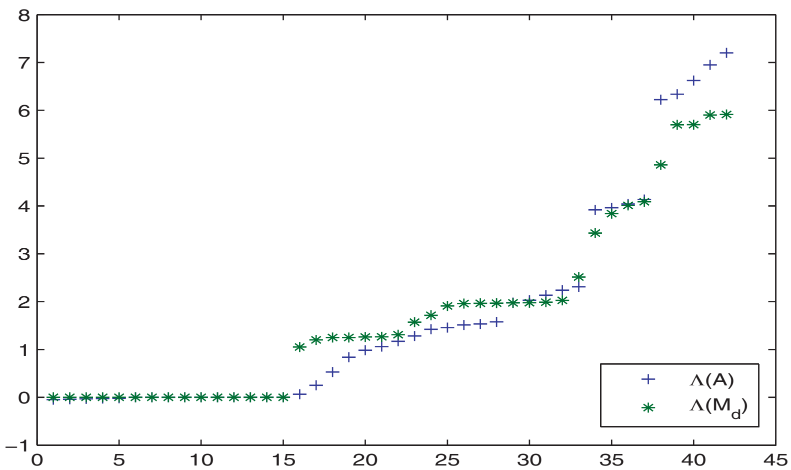

FIG. 4.7. *Eigenvalues of A and M_d of Example* 3.

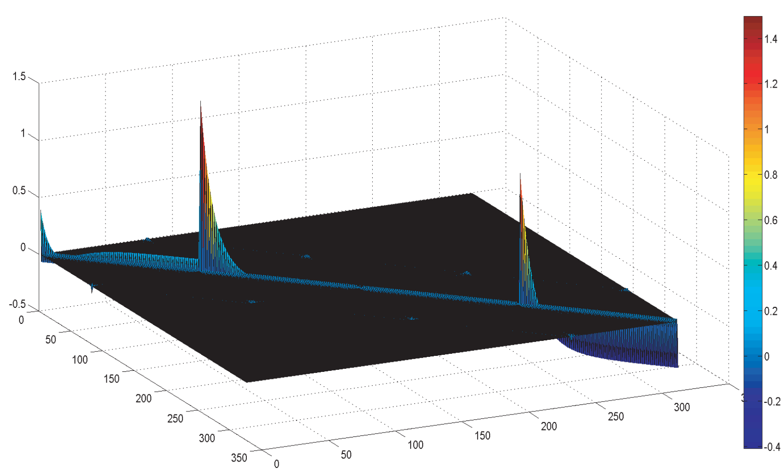

FIG. 4.8. *Plot of the entries of matrix A of Example* 4.

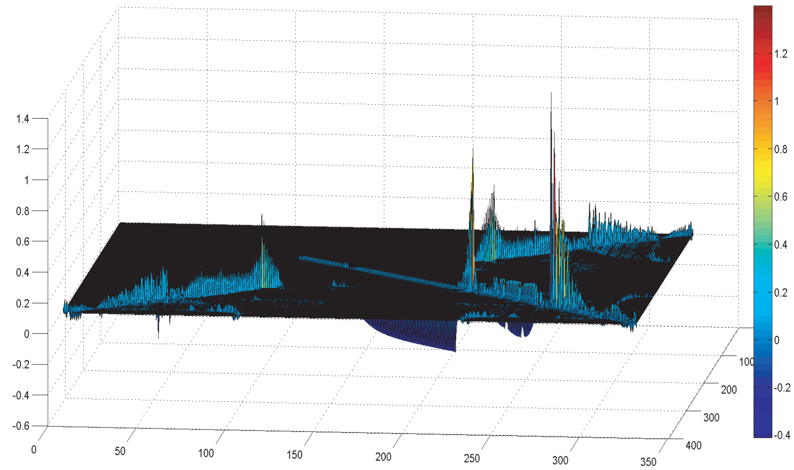

FIG. 4.9. *Plot of the entries of matrix M of Example* 4.

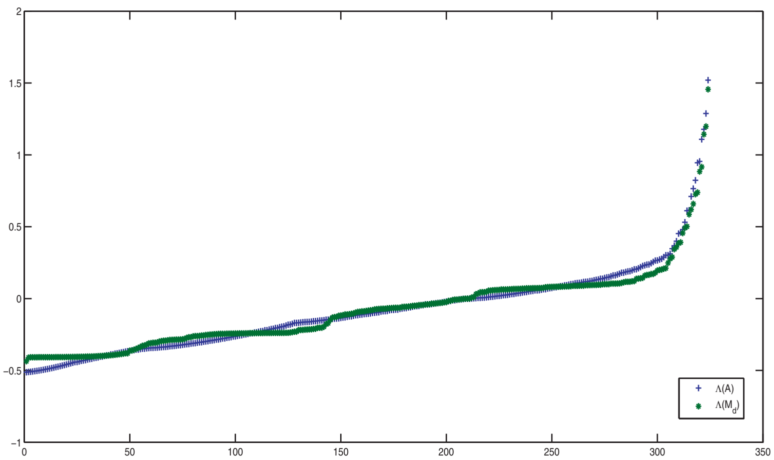

FIG. 4.10. *Eigenvalues of A and M_d of Example* 4.

and it has 211 negative and 113 positive eigenvalues, respectively. The matrix M computed by the proposed algorithm is depicted in Figure 4.9. The eigenvalues of A and M_d are depicted in Figure 4.10. The eigenvalues of M_d approximate well the eigenvalues of A because A is strongly diagonal dominant. The backward error is $\|A - QMQ^T\|_2 = 7.54 \times 10^{-15}$.

5. Conclusions. A new symmetric factorization $A = QMQ^T$, where Q is orthogonal and M is block lower antitriangular, has been proposed. Since Q is orthogonal, the eigenvalues of A and M are the same. The sizes of the blocks of M are shown to yield the inertia of A, and the diagonal elements of some of the blocks are shown to yield, for the considered examples, good estimates of the eigenvalues of A.

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REFERENCES

[1] J.O. Aasen, *On the reduction of a symmetric matrix to tridiagonal form*, BIT, 11 (1971), pp. 233–242.

[2] A.A. Anda and H. Park, *Fast plane rotations with dynamic scaling*, SIAM J. Matrix Anal. Appl., 15 (1994), pp. 162–174.

[3] A.W. Bojanczyk, R.P. Brent, P. Van Dooren, and F. de Hoog, *A note on downdating the Cholesky factorization*, SIAM J. Sci. Statist. Comput., 8 (1987), pp. 210–221.

[4] J.R. Bunch, C.P. Nielsen, and D.C. Sorensen, *Rank-one modification of the symmetric eigenproblem*, Numer. Math., 31 (1978), pp. 31–48.

[5] J.R. Bunch and B.N. Parlett, *Direct methods for solving symmetric indefinite systems of linear equations*, SIAM J. Numer. Anal., 8 (1971), pp. 639–655.

[6] J.R. Bunch and L. Kaufman, *Some stable methods for calculating inertia and solving symmetric linear systems*, Math. Comp., 31 (1977), pp. 163–179.

[7] G.H. Golub, *Some modified matrix eigenvalue problems*, SIAM Rev., 15 (1973), pp. 318–334.

[8] G.H. Golub and C.F. Van Loan, *Matrix Computations*, 3rd ed., The Johns Hopkins University Press, Baltimore, MD, 1996.

[9] I. Gohberg, P. Lancaster, and L. Rodman, *Indefinite Linear Algebra and Applications*, Birkhäuser Verlag, Basel, Switzerland, 2005.

[10] N.I.M. Gould, *On practical conditions for the existence and uniqueness of solutions to the general equality quadratic programming problem,* Math. Program., 32 (1985), pp. 90–99.

[11] W.B. Gragg and W.J. Harrod, *The numerically stable reconstruction of Jacobi matrices from spectral data*, Numer. Math., 44 (1984), pp. 317–335.

[12] S. Hammarling and M.A. Singer, *A canonical form for the algebraic Riccati equation*, in Mathematical Theory of Networks and Systems, P.A. Fuhrmann, ed., Springer-Verlag, Berlin, Germany, 1984.

[13] P.C. Hansen and P. Yalamov, *Computing symmetric rank-revealing decompositions via triangular factorization*, SIAM J. Matrix Anal. Appl., 23 (2001), pp. 443–458.

[14] N.J. Higham, *Accuracy and Stability of Numerical Algorithms*, 2nd ed., SIAM, Philadelphia, 2002.

[15] N.J. Higham, *The Matrix Computation Toolbox*, http://www.ma.man.ac.uk/∼higham/mctoolbox.

[16] D. Kressner, C. Schröder, and D.S. Watkins, *Implicit QR algorithms for palindromic and even eigenvalue problems*, Numer. Algorithms, 51 (2009), pp. 209–238.

[17] D.S. Mackey, N. Mackey, C. Mehl, and V. Mehrmann, *Numerical methods for palindromic eigenvalue problems: Computing the anti-triangular Schur form*, Numer. Linear Algebra Appl., 16 (2009), pp. 63–86.

[18] N. Mastronardi, S. Chandrasekaran, and S. Van Huffel, *Fast and stable algorithms for reducing diagonal plus semi-separable matrices to tridiagonal and bidiagonal form*, BIT, 41 (2001), pp. 149–157.

[19] N. Mastronardi, E. Van Camp, and M. Van Barel, *Divide and conquer algorithms for computing the eigendecomposition of symmetric diagonal–plus–semiseparable matrices*, Numer. Algorithms, 9 (2005), pp. 379–398.

[20] N. Mastronardi and P. Van Dooren, *Recursive approximation of the dominant eigenspace of an indefinite matrix*, J. Comput. Appl. Math., 236, (2012), pp. 4090–4104.

[21] C. Mehl, *On asymptotic convergence of nonsymmetric Jacobi algorithms,* SIAM J. Matrix Anal. Appl., 30 (2008), pp. 291–311.

[22] H. Park and S. Van Huffel, *Two–way bidiagonalization scheme for downdating the singular-value decomposition,* Linear Algebra Appl. 222 (2009), pp. 23–39.

[23] S. Van Huffel and H. Park, *Parallel tri- and bi-diagonalization of bordered bidiagonal matrices*, Parallel Comput., 20 (1994), pp. 1069–1220.