# 3 Lecture 3. Inexact Newton-Krylov methods for solving large nonlinear systems

## 3.1 Plan of the lecture

1. Newton method. Smoothness assumptions.
2. Newton and preconditioned fixed-point iterations.
3. Convergence, important Lemma.
4. The important lemma, residual, error. A stopping criterion.
5. Saving work in the Newton method. Inexact Newton methods.
6. Convergence of inexact Newton methods.
7. Jacobian evaluations. Matrix-free methods.
8. Choice of the linear solver. CGS.

## 3.2 Exercises for Lecture 3

**Exercise 3.1** The following result is called *Banach lemma*:
*If $A$, $B \in \mathbb{R}^{n\times n}$ and $\|I - BA\| < 1$, then $A$ and $B$ are both nonsingular and*

$$\|A^{-1}\| \leqslant \frac{\|B\|}{1 - \|I - BA\|}, \qquad \|A^{-1} - B\| \leqslant \frac{\|B\|\,\|I - BA\|}{1 - \|I - BA\|}.$$

Using the Banach lemma prove the first two statements in the following "important Lemma":
*Let $B(\delta)$ be a ball around $x_*$, $B(\delta) \equiv \{x \mid \|x - x_*\| < \delta\}$. If the assumptions on $F(x)$ as discussed at the lecture hold then $\exists\delta > 0 :\ \forall x \in B(\delta)$*

$$
\begin{aligned}
&(1) \qquad \|F'(x)\| \leqslant 2\|F'(x_*)\|,\\
&(2) \qquad \|F'(x)^{-1}\| \leqslant 2\|F'(x_*)^{-1}\|,\\
&(3) \qquad \frac{1}{2}\|F'(x_*)^{-1}\|^{-1}\|e\| \leqslant \|F(x)\| \leqslant 2\|F'(x_*)\|\|e\|, \quad \textit{where } e = x - x_*.
\end{aligned}
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**Exercise 3.2** We solve linear system $Ax = b$ and write it as $F(x) = 0$, where $F(x) = b - Ax$. Consider the fixed-point iteration for $K(x) = F(x) + x$ and write it down in terms of $A$, $x$, and $b$. Does this method look familiar to you? ◊

**Exercise 3.3** Write down the fixed-point iteration for $K(x) = x - F(x)$ and compare it with the Newton method. What if $F'(x_c) = I$? ◊

**Exercise 3.4** Let $r_m$ be the residual of the Jacobian linear system in the inexact Newton method, with $m$ being the linear iteration number. Assume that the inner iterative linear solver has zero initial guess $s_0 = 0$. What does the stopping criterion of the linear solver discussed at the lecture mean for $\|r_m\|$ and $\|r_0\|$? ◊