A FURTHER EXERCISE

GIVEN $a(z)=\sum_{k=0}^{+\infty} a_k z^k$, SET $\hat{a}(z) = a(zt)a(zt^2)\cdots a(zt^{r-1})$

WHERE $t=\sqrt[r]{1}$ PRINCIPAL

(i.e. $t^r=1$ and $t^i \neq 1$, $0<i<r$).

PROVE THAT $\hat{a}(z)a(z) = a_0^{(1)} + a_1^{(1)} z^r + a_2^{(1)} z^{2r} + \dots = \sum_{k=0}^{\infty} a_k^{(1)} z^{kr} =: a^{(1)}(z)$

FOR SOME $a_i^{(1)}$

($a(z)$ "FULL" $\to$ $a^{(1)}(z)$ "SPARSE").

MOREOVER, PROVE THAT THE COEFFICIENTS OF $\hat{a}(z)$ ARE REAL IF THE COEFFICIENTS OF $a(z)$ ARE REAL.

II LECTURE — III LECTURE — IV LECTURE

REFERENCES: toe_1m (pp. 1-7), toe_1d ($\dots \sqrt{\frac{2}{n+1}} \sin \frac{ij\pi}{n+1} \dots$), genova (pp. 10-), toe_1m (pp. 8-), genova (pp. 1-5, 6-8)

in www.uniroma2.it/~difiore

AND

NOTES OF ANY STUDENT WHO HAS ATTENDED MY LECTURES.
NOTES ABOUT MY FIRST LECTURE ARE NECESSARY.

(1) Let $G=\{1,2,\dots,n\}$ be a group with 1 as identity element.
- Prove that $\mathcal{L}=\{A\in\mathbb{C}^{n\times n}: a_{ij}=a_{ki,kj}\ \ i,j,k\in G\} = \{A\in\mathbb{C}^{n\times n}: a_{i,j}=a_{1,i^{-1}j},\ \forall i,j\in G\}$ is a matrix algebra.
- Is it true that $A\in\mathcal{L},\ \det(A)\neq 0 \Rightarrow A^{-1}\in\mathcal{L}$?
- Let G be the dihedral group of cardinality six. Write the generic matrix of the corresponding group matrix algebra.

(2) Let $\mathcal{C}$ be the set of all $n\times n$ circulant matrices.
- Prove the identity:

$$\mathcal{C}+J\mathcal{C}=\{A\in\mathbb{C}^{n\times n}: AX=XA\},\quad J=\begin{bmatrix}&&&1\\&&⋰&\\&1&&\\1&&&\end{bmatrix},\quad X=\begin{bmatrix}0&1&&&1\\1&0&1&&\\&1&\ddots&\ddots&\\&&\ddots&&1\\1&&&1&0\end{bmatrix}$$

- Is the set $\mathcal{C}+J\mathcal{C}$ commutative?
- Write the generic matrix of the set $\{p(X)\}$ (p = polynomial)

(3) Given $X\in\mathbb{C}^{n\times n}$ and p = polynomial such that $p(X)$ is invertible, is it true that $p(X)^{-1}=q(X)$ for some polynomial q ?

(7) Write explicitly A^{-1}, being $A=\begin{bmatrix}4&1&&0\\1&4&1&\\&1&\ddots&\ddots&1\\0&&1&4\end{bmatrix}$

(hint: note that $A, A^{-1}\in\tau$)

(4) Let $\mathcal{L}$ be the set of lower triangular Toeplitz $n\times n$ matrices.
Is it true that $\mathcal{L}=\{M\Delta M^{-1}: \Delta \text{ diagonals}\}$ for some nonsingular $n\times n$ matrix M ?
Let $\mathcal{L}:=\{M\Delta M^{-1}: \Delta \text{ diagonals}\}$, where M is a $n\times n$ nonsingular matrix. Prove that
$\mathcal{L}=\{p(X): p \text{ polynomials}\}$ for some $X\in\mathbb{C}^{n\times n}$.

(5) Consider the Fourier matrix F,

$$F_{ij}=\frac{1}{\sqrt{n}}\,\omega^{(i-1)(j-1)},\quad i,j=1,\dots,n,\ \text{where}$$

$\omega\ /\ \omega^n=1,\ \omega^i\neq 1\ \ 0<i<n.$

Prove that $F^*F=I$, $F^2=\begin{bmatrix}1&0&\cdots&0\\0&\cdots&0&1\\\vdots&&⋰&\\0&1&&0\end{bmatrix}$.

(6) Write the generic matrix of the set

$$\mathcal{C}_{-1}=\{p(X): p \text{ polynomials}\},$$

$$X=\begin{bmatrix}0&1&&0\\\vdots&&\ddots&\\0&&0&1\\-1&0&\cdots&0\end{bmatrix}.$$

Find a unitary matrix M such that
$\mathcal{C}_{-1}=\{M\Delta M^{-1}: \Delta \text{ diagonals}\}$.