Rome-Moscow School in Numerical Analysis, 2012
Course "Mathematical Methods in Computer Graphics"
Final test

Candidate:

1. Consider a Monte Carlo estimator with N samples, and a stratified method estimator with N samples and N strata, one sample per stratum. Then the variance of the stratified estimator is smaller than the variance of the Monte Carlo estimator ("smaller" means "not larger than"). Prove this statement rigorously (the argument in the book by Dutré, Bala and Bekaert skips some details and in particular does not prove the final inequality). In case $N > 1$, is it true that the variance of the stratified method is *strictly smaller* than the variance of the Monte Carlo method?

2. If we start with an equidistributed random generator with values in $[0, 1)$ and we want to generate an angular distribution on the hemisphere with probability distribution given by the cosine law (proportional to the cosine of the latitude angle, that is the deviation from north pole), we generate pairs (φ, θ) of longitude and latitude angles as follows:

   - *generate a random variable* u_φ *equidistributed in* $[0, 1]$ *and then take* $\varphi = 2\pi u_\varphi$*;*
   - *generate a random variable* u_θ *equidistributed in* $[0, 1]$ *and then take* $\theta = \arccos \sqrt{u_\theta}$.

   Instead, suppose that we want to generate an angular distribution on points (φ, θ) on the hemisphere with joint probability distribution $p(\varphi, \theta)$ proportional to $\cos\theta \, \sin\frac{\varphi}{2}$ (observe that this makes sense because $\sin\frac{\varphi}{2} \geqslant 0$ in $[0, 2\pi]$). In this case, how should we change the previous procedure?

3. To compute a two dimensional integral by the Monte Carlo method, we must generate a random variable on the unit square $[0, 1] \times [0, 1]$. If we use N samples, a natural stratified method is the N-Rooks algorithm, that proceeds as follows:

- *split each side of the square in* N *equal intervals* $I_j = [j/N, (j+1)/N]$*, with* $j = 0, \ldots, N-1$*;*
- *consider the* N^2 *square cells* $K_{ij} = I_i \times I_j$*;*
- *select at random a value of* j_1 *equidistributed in* $\{0, \ldots N-1\}$ *and choose a point* $x_{1j_1} \in K_{1j_1}$ *(with distribution probability equidistributed in this cell) as first sampling point;*
- *exclude the index* $j = j_1$ *and perform the same process on the remaining* $N-1$ *values of* j *to select the second sampling point in a randomly chosen cell* K_{2j_2}*, where automatically* $j_2 \neq j_1$*;*
- *exclude also* $j = j_2$ *and iterate the process until* N *sampling points have been chosen, no two of which belong to cells that share the same row or column.*

However, sometimes in Computer Graphics we need four dimensional integrals, for instance when computing form factors via their definition. In such cases we might choose at random a point from each of the N^4 cells of the lattice decomposition of the unit cube in $\mathbb{R}^4$. But for a stratified sampling method, we need to select at random multi-indices such that no two of them coincide in any of their four digits. How could we proceed?