CONE PRESERVING MATRICES

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The following note is no more than a very brief introduction to what can be called the *Cone Theoretic approach to matrix theory*. In what follows we would like to give an overview of the main concepts which are involved in such theory in order to state and prove, maybe only partially, a generalized version of the famous Perron-Frobenious theorem, which can be attributed to the names of Krein, Rutman, Nusbaum, Birkoff and Vandergraft. Such theorem gave rice to a wide literature during 70s, 80s, and 90s, mainly due to Barker, Schneider, Rothblum and Tam. In particular, we suggest a quite recent survey by Tam [Tam01] to interested readers. Throughout this note we will use the following notations

- $\mathcal{M}_n$ is the set of $n \times n$ matrices
- $\sigma(A)$ is the spectrum of $A \in \mathcal{M}_n$
- $\rho(A)$ is the spectral radius of $A \in \mathcal{M}_n$
- $B_r(c)$ is the open ball centered in c with radius r (the space where the ball lives will be clear from the context). We omit c when it coincides with the origin, i.e. $B_r = B_r(0)$
- $(\,\cdot\,,\,\cdot\,)$ denotes the scalar product

1 Cones

We assume that V is a real finite dimensional vector space such that $\dim V = n$. Given a finite dimensional vector space V, a cone $C \subseteq V$ is a subset such that $C + C \subseteq C$, $C \cap (-C) = \{0\}$ and $\alpha C \subseteq C$, for any $\alpha \geq 0$. We let $\operatorname{int} C$ denotes the larger open subset of C (the interior of C) and $\partial C = C \setminus \operatorname{int} C$ (the boundary of C). If $\operatorname{int} C \neq \emptyset$ we say that C is **full**. In the finite dimensional case a full cone is also **reproducing**, that is $\operatorname{Span} C = C - C = V$, where $\operatorname{Span} C$ is the smaller vector space which contains C. We let $\dim C = \dim \operatorname{Span} C$. A **proper** cone is a closed, convex and full cone.

If C is a closed cone, $F \subseteq C$ is a **face** of C if F is a subcone and if

$$x, y \in C : x + y \in F \Longrightarrow x, y \in F$$

Exercise 1 Prove that if $F \trianglelefteq C$ is nontrivial then $F \in \partial C$.

The set of all faces of C is denoted by $\mathcal{F}(C)$. If $F \in \mathcal{F}(C)$ we write briefly $F \trianglelefteq C$. If $M \subseteq C$ we write $\Phi(M)$ for the least face of C containing M, namely

$$\Phi(M) = \cap \{F \mid F \supseteq M, F \trianglelefteq C\}$$

If $F \in \mathcal{F}(C)$ is one dimensional, then F is an **extremal ray** of C. The set of all the extremal rays of C is denoted by $\operatorname{Ext}(C)$. Any convex cone C is the convex hull of $\operatorname{Ext}(C)$. A cone is **polyhedral** iff it has a finite set of extremal rays. A polyhedral cone is **simplicial** iff $|\operatorname{Ext}(C)| = \dim V$. If V^* is the dual space of V, we let

$$C^* = \{f \in V^* \mid f(x) \geq 0, \forall x \in C\}$$

be the dual cone of C. It is easy to check that C^* is indeed a cone itself and, of course, if $C^* = C$, C is called self dual cone.

THEOREM 1.1 A cone C is polyhedral if and only if it is the intersection of a finite set of half spaces.[1]

THEOREM 1.2 For any polyhedral cone C, C^* is polyhedral and $C^{**} = C$.

From now on we assume that any cone C is a proper cone. If a base for V is fixed, any linear transformation from V to itself is uniquely identified by a $n \times n$ matrix. Therefore we make no distinction between the space of linear transformation from V to itself and the space $\mathcal{M}_n$ of $n \times n$ matrices[2]. Let

$$\Pi(C) = \{A \in \mathcal{M}_n \mid AC \subseteq C\}$$

be the set of matrices which leave C invariant.

Exercise 2 Prove that

1. $\Pi(C)$ is a proper cone of $\mathcal{M}_n$
2. $\operatorname{int} \Pi(C) = \{A \in \mathcal{M}_n \mid A(C \setminus \{0\}) \subseteq \operatorname{int} C\}$
3. $A \in \Pi(C) \iff A^T \in \Pi(C^*)$

EXAMPLE 1.3 A very common example of cone is the so called *ice cream cone* or *Lorentz cone.* It is the cone $\mathcal{K}_p$ defined over $\mathbb{R}^n$ as

$$\mathcal{K}_p = \{x \in \mathbb{R}^n \mid \|(x_1, \dots, x_{n-1})\|_p \leq x_n\}$$

It is a proper cone, its non-trivial faces are precisely its extreme rays, each generated by a nonzero boundary vector, that is one for which the equality holds above. Moreover, it is easy to see that $\mathcal{K}_p^* = \mathcal{K}_p$ if and only if $p = 2$.

Another example is the *nonegative orthant.* It is the cone of all those vectors which have nonnegative entries. We denote such cone by $\mathbb{R}^n_+$. It is a self dual simplicial proper cone, its interior is of course the set of strictly etrywise positive vectors and its set of extremal rays can be explicitly wrote, indeed we easily see that $\operatorname{Ext}(C) = \{e_1, \dots, e_n\}$[3]

The last cone we would like to mention is the cone $\mathcal{S}_n^+$ of *symmetric positive semidefinite matrices.* We consider such cone embedded into $\mathcal{S}_n$, the real space of symmetric matrices. It is proper and selfdual, its interior is the set of positive definite matrices and the set of its extremal rays is the set of symmetric rank one matrices, ie $\operatorname{Ext}(\mathcal{S}_n^+) = \{xx^T \mid x \in \mathbb{R}^n\}$. The interested reader could refer to [HW87] for a survey on such a cone.

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[1] We recall that an half space is the set of points of V which lie onto either of the two parts in which V is divided by an hyperplane. Any $\phi \in V^*$ induces two half spaces $\mathcal{P}_\phi^\pm = \phi^{-1}(\mathbb{R}_\pm)$ and of course for any half space $\mathcal{P}$ we can find $\phi \in V^*$ such that $\mathcal{P} = \mathcal{P}_\phi^+$and $\mathcal{P} = \mathcal{P}_\phi^-$. If V is an Hilbert space, we can describe any half space as the set of all vectors $x \in V$ such that $(x, y) \geq 0$ (or $(x, y) \leq 0$), for a fixed $y \in V$.

[2] Recall that V is defined to be a real vector space, therefore any matrix is assumed to be real throughout this notes. Nevertheless several results we present may be generalized to complex vector spaces.

[3] e_i is the i-th canonical vector $(e_i)_k = \delta_{ik}$.

Any proper cone induces a partial order over V, given by

$$x \succeq 0 \iff x \in C$$

and we write $x \succ 0$ iff $x \in \operatorname{int} C$. Therefore the elements of C are also called C-nonnegative and the elements of $\operatorname{int} C$ are called C-positive. This is also an analogy with the standard cone of nonnegative vectors $\mathbb{R}^n$.

Note that many concept we have introduced before can be reformulated in a different (an sometimes more useful) way, by means of the partial order induced by C. For instance we see that

Exercise 3 Prove that

$$F \trianglelefteq C \iff x \in F \text{ and } 0 \preceq y \preceq x \implies y \in F \ . \tag{1}$$

The characterization above let us easily prove the following

Observation 1.4

1. If $x \in C$ then $\Phi(x) = \{y \in C \mid \alpha y \preceq x \text{ for some } \alpha \geq 0\}$.
2. $F \trianglelefteq C$ and $x \in \operatorname{int} F$ if and only if $F = \Phi(x)$.

Proof. 1. Let $M = \{y \in C \mid \alpha y \preceq x \text{ for some } \alpha \geq 0\}$. It is clear from (1) that $M \trianglelefteq C$ and that $x \in M$. Therefore $M \supseteq \Phi(x)$. On the other hand if $z \in M$, then $\alpha z \preceq x$ and (1) implies $\alpha z \in \Phi(x)$, hence $z \in \Phi(x)$ showing that $M \subseteq \Phi(x)$. 2. Assume that $x \in \operatorname{int} F$, $F \trianglelefteq C$. This implies that for any $y \in F$ there exists $\varepsilon > 0$ such that $x - \varepsilon y \in \operatorname{int} F$, hence $x \succeq \varepsilon y$ i.e. $y \in \Phi(x)$. Thus $F \subseteq \Phi(x)$, the reverse inclusion is obvious. Viceversa if $F = \Phi(x)$ then clearly $F \trianglelefteq C$ and $x \in \operatorname{int} F$, in fact if $x \in \partial F$ then there exists $y \in F$ s.t. for any $\varepsilon > 0$, $x - \varepsilon y \notin C$ (since $\partial F \subseteq \partial C$) ie. $y \notin \Phi(x)$. □

To any proper cone C is associated a norm. Given any $x \in \operatorname{int} C$ consider the order interval $B_x(C) = \{y \in V \mid -x \preceq y \preceq x\}$. One easily observe that $B_x(C)$ is a symmetric convex body, that is $B_x(C)$ is a closed convex symmetric[4] set with nonempty interior. Note that if $z \in B_x(C)$ then the whole set $\{\lambda z \mid |\lambda| \leq 1\}$ belongs to $B_x(C)$. Moreover one easily sees that for any $y \in V$ there exists $\lambda \geq 0$ such that $y \in \lambda B_x(C)$. As a consequence we can define

$$\|y\|_x = \inf\{\lambda \geq 0 \mid y \in \lambda B_x(C)\}$$

which is a norm on V (the reader might prove this easily) such that $\|x\|_x = 1$ and

(1) $u \preceq v \implies \|u\|_x \leq \|v\|_x$
(2) $\|A\|_x = \sup_{\|y\|_x=1} \|Ay\|_x = \|Ax\|_x$ for any $A \in \Pi(C)$.

In fact if $v \preceq \lambda x$ then $\lambda x - u = (\lambda x - v) + (v - u) \in C$ ie. $u \preceq \lambda x$, and $\lambda x + u = \lambda x - u + (u + u) \in C$ implying $-\lambda x \preceq u$. Therefore $v \in \lambda B_x(C) \Rightarrow u \in \lambda B_x(C)$. In a similar way we see that $Ax \preceq \lambda x \Rightarrow \lambda x - Ay = \lambda x - Ax + A(x - y) \in C$ (since $\|y\|_x = 1 \Rightarrow y \preceq x$), then $\lambda x \succeq Ay$ and $\lambda x + Ay = \lambda x - Ax + A(x + y) \in C$ implying $Ay \succeq -\lambda x$ ie. $\|Ay\|_x \leq \|Ax\|_x$.

We spend few words recalling that it is well known that any convex body B induces a norm defined as above, ie $\|x\|_B = \inf\{\lambda \geq 0 \mid x \in \lambda B\}$, and conversely

[4]A set $\Omega \subseteq V$ is *symmetric* if $x \in \Omega$ implies $-x \in \Omega$.

any norm $\|\cdot\|$ easily defines a convex body $B_{\|\cdot\|} = \{x \mid \|x\| \leq 1\}$ which let us write the norm itself as $\|x\| = \inf\{\lambda \geq 0 \mid x \in \lambda B_{\|\cdot\|}\}$.

2 The cone of C-nonnegative matrices

Let us now observe what $\Pi(\mathbb{R}^n_+)$ is. Since any matrix $A \in \Pi(\mathbb{R}^n_+)$ should map e_i onto a nonnegative vector, we see that A is a nonnegative matrix. On the other hand it is obvious that a nonnegative matrix leaves $\mathbb{R}^n_+$ invariant. Thus, $\Pi(\mathbb{R}^n_+)$ is the cone of entrywise nonnegative matrices. According to such observation, it is clear that the cone $\Pi(C)$ is a way to generalize $\Pi(\mathbb{R}^n_+)$, and we will see that it is somehow the most general way possible. For any $A \in \Pi(\mathbb{R}^n_+)$, the well known Perron-Frobenious theory states many interesting and very useful results. The main of them are summarized by the following

Theorem 2.1 (Perron-Frobenious) Let $A \in \Pi(\mathbb{R}^n_+)$ (i.e. $A \geq O$) and let $\rho(A)$ be its spectral radius. Then

1. $\rho(A) \in \sigma(A)$
2. There exist $x \geq 0$ such that $Ax = \rho(A)x$

If moreover $A \geq O$ is irreducible[5], then

3. $\rho(A) \in \sigma(A)$ is simple and nonzero
4. The eigenvector x in 2 is positive (and unique up to a scalar multiple)

If, finally, $A > O$, then

5. $|\lambda| < \rho(A)$, for any $\lambda \in \sigma(A) \setminus \{\rho(A)\}$.

For a complete proof see for instance [Gan59, Var99].

The main aim of this notes is to provide a very general extension of this important Theorem 2.1, involving matrices of $\Pi(C)$. We need, first of all, to extend the definition of irreducible cone-nonnegative matrices

Definition 2.2 *$A \in \Pi(C)$ is C-irreducible iff, for any $F \trianglelefteq C$ such that $AF \subseteq F$, then either $F = \{0\}$ or $F = C$. A is C-reducible if it is not C-irreducible.*

Exercise 4 Prove that the above definition is well posed. In other words prove that A is C-irreducible for $C = \mathbb{R}^n_+$ if and only if there exists a permutation P such that $P^\mathsf{T}AP$ is a block triangular matrix with square diagonal blocks.

Exercise 5 Prove that

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is a $\mathcal{K}_2$-irreducible matrix, where $\mathcal{K}_2$ is the ice-cream cone over the real space $\mathbb{R}^3$.

An important result from the theory of invariant cones on Banach spaces sates that if a compact operator A leaves a proper cone invariant then $\rho(A)$ is an eigenvalue of A and a corresponding eigenvector lies inside the cone. This is known

[5]Recall that a matrix $A \geq O$ is said to be reducible if there exists a permutation matrix P such that $PAP^\mathsf{T} = \begin{pmatrix} X & O \\ W & Y \end{pmatrix}$ where the diagonal blocks are square matrices. A matrix is said to be irreducible if it is not reducible.

as the Krein-Rutman theorem [KR48] and it also holds more in general for continuous maps [Nus88, Sch71]. A proof for the finite dimensional case, based on the Jordan canonical form, was given by Birkoff [Bir67], who also extended the result, obtaining further properties on $\rho(A)$ which turn out to be also a sufficient condition on a matrix for being cone-nonnegative.

Let $A \in \mathcal{M}_n$ and let $\lambda \in \sigma(A)$. We denote by $\tau(\lambda)$ the size of the largest diagonal block containing λ, in the Jordan canonical form of A.

THEOREM 2.3 If C is a proper cone of V and $A \in \Pi(C)$, then

1. $\rho(A) \in \sigma(A)$
2. if $\lambda \in \sigma(A)$ is such that $|\lambda| = \rho(A)$, then $\tau(\lambda) \leq \tau(\rho(A))$
3. there exists $x \in C$ such that $Ax = \rho(A)x$

Conversely, if properties 1 and 2 hold for a matrix $A \in \mathcal{M}_n$, then there exits a proper cone C such that $A \in \Pi(C)$.

The proof of this theorem is not stated here and will be probably insert into a future appendix. However the interested reader can look for it throughout [Bir67], for instance.

Observe that we get an immediate nontrivial consequence. If $\sigma(A) \subset \mathbb{R}_+$ then $A \in \Pi(C)$ for some cone C (since $\rho(A) \in \sigma(A)$ and no other eigenvalues of A have modulus $\rho(A)$). Therefore, in particular, any symmetric positive semidefinite matrix leaves a cone invariant. We already know that such matrices form a cone, nonetheless this theorem gives us some further information: any $A \in \mathcal{S}_n^+$ belongs to $\Pi(C)$, for some C. Actually it is not difficult to understand who is such C. Let for instance $A = Q\Lambda Q^T \in \mathcal{S}_n^+$, then it easily seen that $Q\mathbb{R}_+^n$ is an invariant cone.

THEOREM 2.4 $A \in \Pi(C)$ is C-irreducible $\iff$ ∂C contains no eigenvectors of A

Proof. Assume that $x \in \partial C$ is an eigenvector of A. For any $y \in \Phi(x)$ we have $x - \alpha y \in C$, hence $A(x-\alpha y) = \lambda x - \alpha Ay \in C \Rightarrow x \succeq \frac{\alpha}{\lambda}Ay \Rightarrow Ay \in \Phi(x)$. Therefore $\Phi(x)$ is a nontrivial invariant face implying that A is C-reducible. Viceversa if $F \trianglelefteq C$ is nontrivial and $AF \subseteq F$, we have that $\mathcal{A} = A\|A\|^{-1}$ is such that $\mathcal{A}(F \cap B_1) \subseteq (F \cap B_1)$. Since $F \cap B_1$ is compact and convex, the Brouwer fixed point implies that A has an eigenvector in F, but this is impossible since $F \subseteq \partial C$. □

LEMMA 2.5 If $x_1, x_2 \in \operatorname{int}(C)$ are eigenvectors of $A \in \Pi(C)$ and λ_1, λ_2 are their eigenvalues, then there exists an eigenvector $x_3 \in \partial C$ with eigenvalue λ_3 such that $\lambda_1 = \lambda_2 = \lambda_3$.

Proof. $\lambda_1, \lambda_2 \geq 0$, since $A \in \Pi(C)$, thus assume that $\lambda_1 \geq \lambda_2$. If $\mu = \|x_1\|_{x_2}$ then $x_3 = \mu x_2 - x_1 \in C$ and in particular $x_3 \in \partial C$ (it follows from the definition of μ). Therefore $Ax_3 = \mu Ax_2 - Ax_1 = \mu\lambda_2 x_2 - \lambda_1 x_1 = \lambda_1(\mu\frac{\lambda_2}{\lambda_1}x_2 - x_1) \in C$, hence $\mu\frac{\lambda_2}{\lambda_1}x_2 - x_1 \in C$ implying $\frac{\lambda_2}{\lambda_1} \geq 1 \Rightarrow \lambda_2 \geq \lambda_1$. Then $\lambda_1 = \lambda_2$ and $Ax_3 = \lambda_1 x_3$. □

Theorem 2.4 and Lemma 2.5 were stated by Vandergraft in [Van68] and are crucial in order to prove the following two theorems on C-irreducible matrices

THEOREM 2.6 $A \in \Pi(C)$ is C-irreducible if and only if A has exactly one eigenvector x in C and, in particular, $x \in \operatorname{int}(C)$.

Proof. Let $A \in \Pi(C)$ be C-irreducible. Theorems 2.3 and 2.4 imply that A has an eigenvector $x \in \operatorname{int}(C)$ and that ∂C contains no eigenvectors of A. Furthermore x is unique due to Lemma 2.5. Conversely if A has a unique eigenvector in $\operatorname{int}(C)$ then ∂C contains no eigenvectors and A is C-irreducible. □

THEOREM 2.7 Let $A \in \Pi(C)$ be C-irreducible, then $\rho(A)$ is simple. Conversely if $A \in \mathcal{M}_n$ such that $\rho(A) \in \sigma(A)$ is simple, then there exists a cone C such that $A \in \Pi(C)$ and A is C-irreducible.

Proof. Assume that $\rho(A)$ is not simple. Then there exist linear independent vectors x_1, x_2 such that $x_1 \in \operatorname{int}(C)$,

$$(i) \qquad Ax_1 = \rho(A)x_1 \ \text{ and } \ Ax_2 = \rho(A)x_2, \ \text{ or}$$

$$(ii) \qquad Ax_1 = \rho(A)x_1 \ \text{ and } \ Ax_2 = \rho(A)x_2 + x_1$$

If (i) holds, then for small enough $\varepsilon > 0$ the vector $x_3 = x_1 + \varepsilon x_2$ still belongs to C and $Ax_3 = \rho(A)x_3$, but this contradicts Theorem 2.6. If (ii) holds, let $\lambda_0 = \inf\{\lambda > 0 \mid \lambda x_1 - x_2 \in C\}$. Then $A(\lambda_0 x_1 - x_2) = \lambda_0 \rho(A)x_1 - \rho(A)x_2 - x_1 = \rho(A)((\lambda_0 - \rho(A)^{-1})x_1 - x_2) \in C$. Again this is an absurd since $\lambda_0 - \rho(A)^{-1} < \lambda_0$. □

Of course if $A \in \operatorname{int}\Pi(C)$, A can not leave any face $F \trianglelefteq C$ invariant. Therefore, as for the cone $\Pi(\mathbb{R}^n_+)$, every C-positive matrix is C-irreducible as well. For such matrices, one more property does hold

LEMMA 2.8 For every complex number ζ which does not lie over the positive real axis, there exist positive numbers $\alpha_1, \dots, \alpha_k$ such that $\sum_{i=1}^{k} \alpha_i \zeta^i = 0$.

Proof. We simply give an informal sketch of the proof. Assume for simplicity that $\zeta = e^{i\theta}$. If θ is a rational multiple of π, the proof is quite simple. So let us consider the general case. The points ζ^k are rotations of ζ along the unit disk and for any pair of them, say ζ^a and ζ^b, we may consider the two dimensional cone $C = \{\alpha\zeta^a + \beta\zeta^b \mid \alpha, \beta \geq 0\}$. Now consider a third point ζ^c. If $\zeta^c \in C$ then it must be a convex combination of ζ^a and ζ^b thus there exist $\alpha_1, \alpha_2, \alpha_3 \geq 0$ such that $\alpha_1\zeta^a + \alpha_2\zeta^b + \alpha_3\zeta^c = 0$. Since the number of powers of ζ we are considering is totally arbitrary, if $\zeta^c \notin C$ we can assume that it lies inside the cone generated by two other powers of ζ. The thesis can be formalized proceeding in this way. □

THEOREM 2.9 If $A \in \operatorname{int}\Pi(C)$ then $|\lambda| < \rho(A)$, for any $\lambda \in \sigma(A) \setminus \{\rho(A)\}$. Conversely if $A \in \mathcal{M}_n$ is such that $\rho(A) \in \sigma(A)$ and $|\lambda| < \rho(A)$ for any $\lambda \in \sigma(A) \setminus \{\rho(A)\}$, then there exists a cone C such that $A \in \operatorname{int}\Pi(C)$.

Proof. We can assume $\rho(A) = 1$ w.l.o.g. Suppose that there exists $\lambda \in \sigma(A) \setminus \{\rho(A)\}$ such that $|\lambda| = \rho(A)$. Then $\lambda = e^{i\theta}$. Let x_λ be its eigenvector and $x \in \operatorname{int} C$ be the eigenvector corresponding to $\rho(A)$. First of all we show that there exists ϕ such that $\operatorname{Re}(e^{i\phi}x_\lambda) \in C$. Let $\mu_\phi = \inf\{\mu > 0 \mid \mu x + \operatorname{Re}(e^{i\phi}x_\lambda) \in C\}$. If $\operatorname{Re}(e^{i\phi}x_\lambda) \notin C$ then $\mu_\phi > 0$ and by definition $y_\phi = \mu_\phi x + \operatorname{Re}(e^{i\phi}x_\lambda)$ lies on the boundary ∂C. Now, since $A \in \operatorname{int}\Pi(C)$ is real, we see that

$$\mu_\phi x + \operatorname{Re}(e^{i(\phi+\theta)}x_\lambda) = Ay_\phi \in \operatorname{int} C$$

therefore or $\operatorname{Re}(e^{i(\phi+\theta)}x_\lambda) \in C$ or $\mu_\phi > \mu_{\phi+\theta} > 0$ (since by definition of $\mu_{\phi+\theta}$, $y_{\phi+\theta}$ is on ∂C). Therefore $\inf_\phi \mu_\phi = \mu_{\phi_0} = 0$ and $y_0 = \operatorname{Re}(e^{i\phi_0}x_\lambda) \in C$.

Now applying Lemma 2.8 to $\lambda = e^{i\theta}$ we see that

$$\sum_{k=0}^{m} \alpha_k \lambda^k = \sum_{k=0}^{m} \alpha_k e^{ik\theta} = 0$$

for positive $\alpha_1, \dots, \alpha_m$. Hence,

$$\sum_{k=0}^{m} \alpha_k A^k y_0 = \sum_{k=0}^{m} \alpha_k \operatorname{Re}(e^{i(\phi_0+k\theta)} x_\lambda) = \operatorname{Re}\left(e^{i\phi_0} x_\lambda \Big(\sum_{k=0}^{m} \alpha_k e^{ik\theta} \Big) \right) = 0$$

and $\sum_k \alpha_k A^k \in \operatorname{int}\Pi(C)$ implies $y_0 = \operatorname{Re}(e^{i\phi_0} x_\lambda) = 0$. Thus we have found that $y_\lambda = ie^{i\phi_0} x_\lambda$ is a real eigenvector of λ, implying that λ is real and equals to -1 (since A is real). Finally let $\omega_0 = \min\{\omega > 0 \mid \omega x + y_\lambda \in C\}$. By definition we have $\omega_0 x + y_\lambda \in \partial C$ and on the other hand $A^2(\omega_0 x + y_\lambda) = \omega_0 x + y_\lambda \in \operatorname{int} C$, obtaining a contradiction which proves the thesis. □

Theorems 2.3, 2.7, 2.9 have been proved partially. They also claim that whenever some PF-like properties hold for a generic matrix A, then $A \in \mathcal{M}_n$ is C-nonnegative, or C-irreducible, or C-positive, for some cone C. The existence of such a cone has been proved by Birkoff [Bir67], under the assumptions of Theorem 2.3, then it has been used just one year later by Vandergraft [Van68] to prove Theorems 2.7, 2.9. Despite they explicitly write such cone C, it has a very unfriendly formulation since it is wrote in terms of eigenvectors and principal vectors of A. The possibility of finding easier formulation of C, at least for some particular cones is, to the best of my knowledge, still a research topic.

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