**Khakim D. Ikramov**

# Exercises for the lectures

1.1.1. Let $F$ be a finite field with $q$ elements. Prove that $x^q - x$ is a nonzero polynomial in $F[x]$ which vanishes at every point of $F$.

1.2.1. The four-leaved rose is the curve defined by the polar equation $r = sin(2\theta)$. (a) Show that the four-leaved rose is contained in the affine variety $\mathrm{V}((x^2+y^2)^3 - 4x^2y^2)$. (b) Prove the converse: $\mathrm{V}((x^2+y^2)^3 - 4x^2y^2)$ is contained in the four-leaved rose.

1.2.2. Prove that the upper half-plane $R = \{(x, y) \in \mathbf{R}^2 : y > 0\}$ is not an affine variety.

1.2.3. Prove that $\mathbf{Z}^n$ (the set of points with integer coordinates in $\mathbf{C}^n$) is not an affine variety.

1.2.4. Give an example to show that an infinite union of affine varieties need not be an affine variety.

1.2.5. Give an example to show that the set-theoretic difference $V - W$ of two affine varieties need not be an affine variety.

1.4.1. Prove the equality of the following ideals in $k[x, y]$:

$$< x + xy, y + xy, x^2, y^2 > = < x, y > .$$

1.4.2. Let $V \subset \mathbf{R}^3$ be the curve parameterized by $(t^2, t^3, t^4)$: (a) Prove that $V$ is an affine variety. (b) Determine $\mathrm{I}(V)$.

1.5.1. Find $\mathrm{GCD}(x^3 + 2x^2 - x - 2, x^3 - 2x^2 - x + 2, x^3 - x^2 - 4x + 4)$. (You may use a computer algebra system.)

1.5.2. Decide whether or not $x^2 - 4 \in < x^3 + x^2 - 4x - 4, x^3 - x^2 - 4x + 4, x^3 - 2x^2 - x + 2 >$.

1.5.3. Let $f_1, \dots, f_s \in \mathbf{C}[x]$. Prove that $\mathrm{V}(f_1, \dots, f_s) = \emptyset$ if and only if $\mathrm{CGD}(f_1, \dots, f_s) = 1$.

1.5.4. Find a basis for the ideal $\mathrm{I}(\mathrm{V}(x^5 - 2x^4 + 2x^2 - x, x^5 - x^4 - 2x^3 + 2x^2 + x - 1))$.

2.2.1. Each of the following polynomials is written with its monomials ordered according to (exactly) one of lex, grlex, or grevlex order. Determine which monomial order was used in each case.

(a) $f(x, y, z) = 7x^2y^4z - 2xy^6 + x^2y^2$. (b) $f(x, y, z) = xy^3z + xy^2z^2 + x^2z^3$. (c) $f(x, y, z) = x^4y^5z + 2x^3y^2z - 4xy^2z^4$.

2.3.1. Compute the remainder on division of the given polynomial $f$ by the ordered set $F$. Use first the grlex order, then the lex order in each case.

(a) $f = x^7y^2 + x^3y^2 - y + 1, F = (xy^2 - x, x - y^3)$. (b) Repeat part (a) with the order of the set $F$ reversed.

2.3.2. Compute the remainder on division:

(a) $f = xy^2z^2 + xy - yz, F = (x - y^2, y - z^3, z^2 - 1)$. (b) Repeat part (a) with the order of the set $F$ permuted cyclically.

2.3.3. Prove that every polynomial $f \in \mathbf{R}[x, y, z]$ can be written as

$$f = h_1(y - x^2) + h_2(z - x^3) + r,$$

where $r$ is a polynomial in $x$ alone.

2.3.4. Find an explicit representation

$$z^2 - x^4y = h_1(y - x^2) + h_2(z - x^3)$$

using the division algorithm.

2.4.1. Let $I \subset k[x_1, \dots, x_n]$ be an ideal with the property that, for every $f = \sum_\alpha c_\alpha x^\alpha \in I$, every monomial $x^\alpha$ appearing in $f$ is also in $I$. Show that $I$ is a monomial ideal.

2.4.2. Let $I =< x^\alpha : \alpha \in A >$ be a monomial ideal with a basis $x^{\beta(1)}, \ldots, x^{\beta(s)}$. Prove that $I =< x^{\alpha(1)}, \ldots, x^{\alpha(s)} >$, where $\alpha(i) \in A$ for $i = 1, \ldots, s$.

2.4.3. Let $I =< x^{\alpha(1)}, \ldots, x^{\alpha(s)} >$ be a monomial ideal. Prove that a polynomial $f$ is in $I$ if and only if the remainder of $f$ on division by $x^{\alpha(1)}, \ldots, x^{\alpha(s)}$ is zero.

2.5.1. Let $I =< g_1, g_2, g_3 > \subset \mathbf{R}[x, y, z]$, where $g_1 = xy^2 - xz + y, g_2 = xy - z^2$, and $g_3 = x - yz^4$. Using the lex order, give an example of $g \in I$ such that $\mathrm{LT}(g) \notin < \mathrm{LT}(g_1), \mathrm{LT}(g_2), \mathrm{LT}(g_3) >$.

2.5.2. Let $I$ be an ideal of $k[x_1, \ldots, x_n]$. Show that $G = \{g_1, \ldots, g_s\} \subset I$ is a Groebner basis of $I$ if and only if the leading term of any element of $I$ is divisible by one of the $\mathrm{LT}(g_i)$.

2.5.3. Is $\{x^4y^2 - z^5, x^3y^3 - 1, x^2y^4 - 2z\}$ a Groebner basis for the ideal generated by these polynomials if the order is grlex with $x > y > z$?

2.5.4. The same question as in Exercise 2.5.3 for the lex order.

2.5.5. Let $I \subset k[x_1, \ldots, x_n]$ be a principal ideal (that is, $I$ is generated by a single $f \in I$). Show that any finite subset of $I$ containing a generator for $I$ is a Groebner basis for $I$.

2.5.6. Prove that, if we take as hypothesis that every ascending chain of ideals in $k[x_1, \ldots, x_n]$ stabilizes, then the conclusion of the Hilbert Basis Theorem is a consequence.

2.5.7. Given polynomials $f_1, f_2, \ldots \in k[x_1, \ldots, x_n]$, let $\mathrm{V}(f_1, f_2, \ldots) \subset k^n$ be the set of solutions to the infinite system of equations $f_1 = f_2 = \cdots = 0$. Show that there is some $N$ such that $\mathrm{V}(f_1, f_2, \ldots) = \mathrm{V}(f_1, f_2, \ldots, f_N)$.

2.5.8. Let $V \subset k^n$ be an affine variety. Prove that $\mathrm{V}(\mathrm{I}(V)) = V$.

2.5.9. Assume that $g \in k[x_1, \ldots, x_n]$ factors as $g = g_1 g_2$. Show that, for any $f$, $\mathrm{V}(f, g) = \mathrm{V}(f, g_1) \cup \mathrm{V}(f, g_2)$.

2.5.10. Show that, in $\mathbf{R}^3$, $\mathrm{V}(y - x^2, xz - y^2) = \mathrm{V}(y - x^2, xz - x^4)$.

2.6.1. Let $G$ be a basis for an ideal $I$ with the property that $f^G = 0$ for all $f \in I$. Prove that $G$ is a Groebner basis for $I$.

2.6.2. Let $G$ and $G'$ be Groebner bases for the ideal $I$ with respect to the same monomial order in $k[x_1, \ldots, x_n]$. Show that $f^G = f^{G'}$ for all $f \in k[x_1, \ldots, x_n]$.

2.6.3. Compute $S(f, g)$ using the lex order for $f = x^7y^2z + 2ixyz, g = 2x^7y^2z + 4$.

2.6.4. Does $S(f, g)$ depend on which monomial order is used? Illustrate your assertion with examples.

2.6.5. Show that $\{y - x^2, z - x^3\}$ is not a Groebner basis for lex order with $x > y > z$.

2.6.6. Determine whether the following sets $G$ are Groebner bases for the ideal they generate:

(a) $G = \{x^2 - y, x^3 - z\}$, grlex order; (b) $G = \{xy^2 - xz + y, xy - z^2, x - yz^4\}$, lex order.

2.6.7. Let $I \subset k[x_1, \ldots, x_n]$ be an ideal, and let $G$ be a Groebner basis of $I$.

(a) Show that $f^G = g^G$ if and only if $f - g \in I$. (b) Deduce that

$$(f + g)^G = f^G + g^G.$$

(c) Deduce that

$$(fg)^G = (f^G \cdot g^G)^G.$$

2.7.1. Find a Groebner basis for the ideal $I = < x^2 + y, x^4 + 2x^2y + y^2 + 3 >$. Use the lex, then the grlex order, and then compare your results.

2.7.2. Find reduced Groebner bases for the ideal in Exercise 2.7.1 with respect to the lex and the grlex orders.

2.7.3. Fix a monomial order, and let $G$ and $\tilde{G}$ be minimal Groebner bases for the ideal $I$. (a) Prove that $\mathrm{LT}(G) = \mathrm{LT}(\tilde{G})$. (b) Conclude that $G$ and $\tilde{G}$ have the same number of elements.

2.7.4. Consider the ideal

$$I =< 3x - 6y - 2z, 2x - 4y + 4w, x - 2y - z - w > \subset k[x, y, z, w].$$

The order is lex with $x > y > z > w$. Find a minimal Groebner basis and the reduced Groebner basis for $I$. Describe their relation to the row echelon matrix and the reduced row echelon matrix obtained from the matrix formed of the coefficients of the original generators of $I$.

2.8.1. Determine whether or not $f = xy^3 - z^2 + y^5 - z^3$ is in the ideal $I =< -x^3 + y, x^2y - z >$.

2.8.2. The same question as in Exercise 2.8.1 for $f = x^3z - 2y^2$ and $I =< xz - y, xy + 2z^2, y - z >$.

2.8.3. Using the Groebner basis technique, find all the critical points of the function

$$f(x, y) = (x^2 + y^2 - 4)(x^2 + y^2 - 1) + (x - 3/2)^2 + (y - 3/2)^2.$$

2.8.4. Solve the following systems in $\mathbf{C}(x, y, z)$:

(a)

$$xy^2 - z - z^2 = 0,$$

$$x^2y - y = 0,$$

$$y^2 - z^2 = 0.$$

(b)

$$xy + z - 1 = 0,$$

$$x - y - z^2 = 0,$$

$$x^2 - 2y + 1 = 0.$$

(c)

$$zx - y - x + xy = 0,$$

$$yz - z + x^2 + yx^2 = 0,$$

$$x - x^2 + y = 0.$$

(d)

$$xy - xz + y^2 = 0,$$

$$yz - x^2 + x^2 y = 0,$$

$$x - xy + y = 0.$$

(e)

$$x^2 + z^2 y + yz = 0,$$

$$y^2 - zx + x = 0,$$

$$xy + z^2 - 1 = 0.$$