Introduction to Matrix Equations

Problems

Problem 1. Find the inverse of the matrix

$$\begin{pmatrix} 1 & -2 & 2 & -4 \\ -2 & 3 & -4 & 6 \\ 3 & -6 & 5 & -10 \\ -6 & 9 & -10 & 15 \end{pmatrix}.$$

Hint. Use the properties of the Kronecker products.

Problem 2. Solve the matrix equation

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + X \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Problem 3. Using a finite computation involving only arithmetic operations, find out which of the matrices

$$A = \begin{pmatrix} 3 & 1 & -1 \\ -3 & -1 & 3 \\ -2 & -2 & 4 \end{pmatrix}, B = \begin{pmatrix} 5 & 5 & -2 \\ -2 & -1 & 1 \\ -1 & -1 & 2 \end{pmatrix}, \quad \text{and} \quad C = \begin{pmatrix} 6 & 0 & 8 \\ 3 & 2 & 6 \\ -2 & 0 & -2 \end{pmatrix}$$

are similar.

Problem 4. Nonderogatory matrices are complex matrices with each eigenvalue having the geometric multiplicity 1. Prove that a complex $n \times n$ matrix A is nonderogatory if and only if the minimal polynomial of A is identical to its characteristic polynomial.

Problem 5. Based on Problem 4, propose a finite algorithm, involving only arithmetic operations, for verifying whether a given $n \times n$ matrix A is nonderogatory.

Problem 6. Let

$$A = \begin{pmatrix} 11 & 4 \\ -4 & 3 \end{pmatrix}.$$

Describe all the solutions to the equation $AX = XA^T$. Are they symmetric matrices? How do you explain this?

Problem 7. Apply the Bartels–Stewart algorithm in order to solve the Sylvester matrix equation

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} X - X \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Problem 8. Apply the Bartels–Stewart algorithm in order to solve the Stein matrix equation

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} X \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} - X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$