As a result of Stages (a) and (b), the original Eq. (1) is replaced by the new Sylvester equation

$$RY - YS = D, \quad Y = U^*XV. \qquad (40)$$

(c) Solving Eq. (40).

Setting

$D = (d_1\, d_2 \dots d_n)$, $Y = (y_1\, y_2 \cdots y_n)$, $d_j, y_j \in \mathbb{C}^n$ $(j = 1, 2, \dots, n)$, we derive from (40) the relation

$$Ry_1 - s_{11}y_1 = d_1. \qquad (41)$$

This is an upper triangular system of linear equations. The diagonal entries of the matrix $R - s_{11}I_m$ are the differences $\lambda_i - \mu_1$ $(i = 1, 2, \dots, m)$, which are nonzero according to (8). Thus, system (41) determines a unique vector y_1. Now that y_1 is known, we derive from (40) the relation

$$Ry_2 - s_{22}y_2 = d_2 + s_{12}y_1,$$

which uniquely determines the vector y_2. Continuing in this way, we find the entire matrix Y.

(d) Returning to the original matrix X.

Inverting the second relation in (40), we obtain

$$X = UYV^*. \qquad (42)$$

Problem 13. Apply the Bartels - Stewart algorithm to solve the matrix equation

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix} X - X \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = I_3 .$$

Hint: $B = A + 3 I_3$. Вставка на третьей странице обложки

Answer: $X = -\frac{1}{3} I_3$.

The discrete Sylvester equation (also called the Stein equation)

This is an equation of the form

$$AXB - X = C. \qquad (43)$$

Again, A and B are given square matrices of possibly different orders m and n, while a given matrix C and the unknown matrix X are of size $m \times n$.

If either A or B is nonsingular, then (43) can be reduced to an equivalent continuous equation. For instance, assume that $\det B \neq 0$. Then (43) is equivalent to the equation

$$AX - X\tilde{B} = \tilde{C},$$

where

$$\tilde{B} = B^{-1}, \quad \tilde{C} = CB^{-1}.$$

Since we allow for the case where both A and B are singular, we do not use this reduction.

Let us derive conditions for the unique solvability of Eq. (43). This is equivalent to the requirement that the homogeneous equation

$$X = AXB \tag{44}$$

have only the trivial solution.

As was the case with the continuous equations, we resort to the Jordan form method. Similarly to (10)-(12), we assume that

$$A = UJ_A U^{-1}, \qquad B = VJ_B V^{-1}, \tag{45}$$

where

$$J_A = J_{p_1}(\lambda_1) \oplus J_{p_2}(\lambda_2) \oplus \cdots \oplus J_{p_u}(\lambda_u) \tag{46}$$

$$(p_1 + p_2 + \cdots + p_u = m),$$

$$J_B = J_{q_1}(\mu_1) \oplus J_{q_2}(\mu_2) \oplus \cdots \oplus J_{q_v}(\mu_v) \tag{47}$$

$$(q_1 + q_2 + \cdots + q_v = n).$$

Upon substitution of (45) into (44), we have

$$X = UJ_A U^{-1} X V J_B V^{-1}$$

or

$$U^{-1}XV = J_A(U^{-1}XV)J_B. \tag{48}$$

Define

$$Y = U^{-1} X V. \qquad (49)$$

Then (48) takes the same form

$$Y = J_A \, Y \, J_B \qquad (50)$$

as the original Eq. (44). We again partition Y into blocks in accordance with J_A and J_B:

$$Y = (Y_{\alpha\beta}), \quad \alpha = 1, 2, \dots, u; \ \beta = 1, 2, \dots, v. \qquad (51)$$

Then Eq. (50) splits into uv separate matrix equations of the form

$$Y_{\alpha\beta} = J_{p_\alpha}(\lambda_\alpha) \, Y_{\alpha\beta} \, J_{q_\beta}(\mu_\beta). \qquad (52)$$

We drop the indices and consider the equation

$$Y = J_k(\lambda) \, Y \, J_\ell(\mu). \qquad (53)$$

Our goal is to find a criterion for Eq. (53) to have only the trivial solution. Consider first the case where either λ or μ is nonzero. For definiteness, assume that $\mu \neq 0$. Then (53) can be rewritten as

$$J_k(\lambda) \, Y = Y \left(J_\ell(\mu)\right)^{-1}. \qquad (54)$$

It is easy to verify that the Jordan form of the matrix $\left(J_\ell(\mu)\right)^{-1}$ is a single Jordan block of dimension ℓ

 corresponding to μ^{-1}. Thus, the homogeneous continuous equation (54) has only the trivial solution if and only if

$$\mu^{-1} \neq \lambda$$

or

$$\lambda\mu \neq 1. \qquad (55)$$

We arrive at the same conclusion if $\mu = 0$ but $\lambda \neq 0$.

Now assume that $\lambda = \mu = 0$, that is,

$$Y = J_k(0)\, Y\, J_\ell(0). \qquad (56)$$

This implies that

$$Y = J_k^2(0)\, Y\, J_\ell^2(0) = J_k^3(0)\, Y\, J_\ell^3(0) = \cdots = J_k^m(0)\, Y\, J_\ell^m(0), \quad m = 4, 5, \ldots$$

Since $J_k^k(0) = 0$ and $J_\ell^\ell(0) = 0$, we conclude that Eq. (56) always has only the trivial solution. Note that condition (55) is fulfilled if $\lambda = \mu = 0$.

We summarize the above analysis as follows: Eq. (43) is uniquely solvable for every right-hand side C if and only if

$$\lambda_i(A)\, \lambda_j(B) \neq 1 \quad \forall i, j. \qquad (57)$$

Solving Eq. (43) numerically

The Bartels-Stewart algorithm can be adapted for the discrete Sylvester equation (43). Stages (a) and (b) are the same as for the continuous case (see formulas (38) and (39)). They result in that Eq. (43) is replaced by a new equation of the same type:

$$RYS - Y = D, \qquad Y = V^{*} X V. \tag{58}$$

At stage (c), we again set

$$D = (d_1\, d_2 \ldots d_n), \quad Y = (y_1\, y_2 \ldots y_n), \quad d_j, y_j \in \mathbb{C}^n \ (j = 1, 2, \ldots, n).$$

Then, equating the first columns in (58), we obtain

$$s_{11} R y_1 - y_1 = d_1. \tag{59}$$

This is an upper triangular system of linear equations. The diagonal entries of the matrix $s_{11} R - I_m$ are the values $\mu_1 \lambda_i - 1$ $(i = 1, 2, \ldots, m)$. Assuming that conditions (57) are fulfilled, we conclude that $s_{11} R - I_m$ is a nonsingular matrix and system (59) determines a unique vector y_1. Now that y_1 is known, we derive from (58) the relation

$$s_{22} R y_2 - y_2 = d_2 - s_{12} R y_1,$$

which uniquely determines the vector y_3. Continuing in the same way, we find the entire matrix Y.

Stage (d) is again as before. The estimates for the costs of the individual stages indicated for the continuous case remain valid here as well.

Problem 14. Apply the Bartels-Stewart algorithm to solve the equation

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} X \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix} - X = I_3 .$$

Hint: $B = 4A^{-1}$

Answer: $X = \frac{1}{3} I_3$.

The equation $AX - X^T B = C$

This equation looks very much like to the continuous Sylvester equation. However, the two equations differ in a number of respects. First, the matrix coefficients A and B can be rectangular matrices. Assume that A is an $m \times n$ matrix. Then, for the products AX and $X^T B$ to have the same size, X and B

must be $n \times m$ matrices, which implies that C is a square matrix of order m.

Even if we assume that $m = n$, the equation

$$AX - X^T B = C \qquad (60)$$

is still very different from (1). For instance, set

$$A = I_n, \quad B = -I_n.$$

Then (1) becomes

$$X = \frac{1}{2} C;$$

that is, Eq. (1) is uniquely solvable for every C. On the other hand, (60) becomes

$$X + X^T = C. \qquad (61)$$

Since the left-hand side is symmetric, C must be symmetric as well. If this condition is fulfilled, then Eq. (61) is solvable; however, it has infinitely many solutions. Namely, the solutions to (61) are described by the formula

$$X = \frac{1}{2} C + K,$$

where K is an arbitrary skew-symmetric matrix.

To simplify the Sylvester equations, we used similarity transformations for the matrices A and B.

(34) Since A and B may now be rectangular, similarity transformations are in general not applicable. Let us see what happens to Eq. (60) if we change the variable according to the formula

$$X = PYQ, \qquad (62)$$

where

$$P \in M_n, \quad Q \in M_m, \quad \det P \neq 0, \quad \det Q \neq 0.$$

Substituting (62) into (60), we obtain

$$APYQ - Q^T Y^T P^T B = C$$

or

$$(Q^{-T} A P) Y - Y^T (P^T B Q^{-1}) = Q^{-T} C Q^{-1}. \qquad (63)$$

Setting

$$\tilde{A} = Q^{-T} A P, \quad \tilde{B} = P^T B Q^{-1}, \quad \tilde{C} = Q^{-T} C Q^{-1},$$

we see that (63) is an equation of the same type as the original equation (60):

$$\tilde{A} Y - Y^T \tilde{B} = \tilde{C}. \qquad (64)$$

Note that, in the transition from (60) to (64), the matrix coefficients A and B are modified according to the formulas

$$A \to \tilde{A} = Q^{-T} A P, \quad B^T \to \tilde{B}^T = Q^{-T} B^T P. \qquad (65)$$

These are not similarity transformations. Formulas (35) (65) motivate the following definition.

Let (E, F) and $(\tilde{E}, \tilde{F})$ be two pairs of $m \times n$ matrices. These pairs are said to be equivalent if

$$\tilde{E} = PEQ, \quad \tilde{F} = PFQ \tag{66}$$

for some nonsingular matrices $P \in M_m$ and $Q \in M_n$.

We distinguish between two types of matrix pairs. A pair (E, F) is said to be regular if $m = n$ and the determinant

$$f(\lambda) = \det (E - \lambda F) \tag{67}$$

does not vanish identically with respect to λ. Otherwise, (E, F) is called a singular pair.

We derive unique solvability conditions for Eq. (60) in the case where the pair (A, B^T) is regular. The more difficult case where this pair is singular is treated in a recent paper Kh. D. Ikramov and Yu. O. Vorontsov // Doklady Mathematics 83 (3) (2011) 380-383.

In our analysis of the Sylvester equations, we used such a powerful tool as the reduction to Jordan form. We need a comparable tool for the pair (A, B^T) with similarity transformations replaced by equivalence ones. Consequently, we consider the following question: what is a pair of the simplest form in the equivalence class of the given regular pair (E, F)?

We first note that there are scalar invariants of equivalence transformations analogous to the eigenvalues of a matrix. These are the roots of polynomial (67), which are called the (finite) eigenvalues of the pair (E, F).

<u>Problem 15.</u> Prove that $\det(E - \lambda F)$, where the matrices $E, F \in M_n$ constitute a regular pair, is invariant under equivalence transformations.

Problem 15 explains why polynomial (67) is called the characteristic polynomial of the pair (E, F).

There is an important distinction between the characteristic polynomials of a matrix $A \in M_n$ and a pair $(E, F) \in M_n \times M_n$. The former is always of degree n, while the degree of the latter can be less than n and even zero. For instance, if $E = I_2$ and $F = J_2(0)$, then $\det(E - \lambda F) \equiv 1$.

Let us agree that, if

$$m = \deg f(\lambda) < n,$$

then, along with m finite eigenvalues, the pair (E, F) has also the eigenvalue ∞ of multiplicity $n-m$. (There are good reasons for this convention, but, unfortunately, we have no time to dwell on these reasons.)

Now we can formulate an important theorem, due to Weierstrass, which plays the same role in the equivalence theory of matrix pairs as the Jordan theorem does in the similarity theory of matrices.

The Weierstrass theorem. Let (E, F) be a regular pair of $n \times n$ matrices. Then there exist nonsingular matrices $P, Q \in M_n$ such that the matrices $\tilde{E}$ and $\tilde{F}$ defined by (66) have the following block diagonal form:

$$\tilde{E} = J_{m_1} \oplus \cdots \oplus J_{m_f} \oplus I_{n_1} \oplus \cdots \oplus I_{n_i}, \qquad (68)$$

$$\tilde{F} = I_{m_1} \oplus \cdots \oplus I_{m_f} \oplus J^{\circ}_{n_1} \oplus \cdots \oplus J^{\circ}_{n_i}. \qquad (69)$$

Here, $J_{m_1}, \ldots, J_{m_f}$ are Jordan blocks of orders $m_1, \ldots, m_f$, respectively, associated with the finite eigenvalues of the pair (E, F); $J^{\circ}_{n_1}, \ldots, J^{\circ}_{n_i}$ are the Jordan blocks (with zero on the principal diagonal) of orders $n_1, \ldots, n_i$; and $I_{m_1}, \ldots, I_{m_f}, I_{n_1}, \ldots, I_{n_i}$ are the identity matrices of the corresponding orders. Furthermore,

$$m_1 + \cdots + m_f + n_1 + \cdots + n_i = n.$$

There are no blocks $J^0_{n_1}, \ldots, J^0_{n_i}$ in (65) if F is a nonsingular matrix. If E is nonsingular, then there is no zero among the finite eigenvalues of the pair (E, F).

We return to the analysis of Eq. (60). It is uniquely solvable if and only if the homogeneous equation

$$AX = X^T B \qquad (70)$$

has only the trivial solution. We first show that, for this to be true, at least one of the matrices A and B must be nonsingular.

Suppose that both A and B are singular matrices. Apply the Weierstrass theorem to the pair (A, B^T), and let $(\tilde{A}, \tilde{B}^T)$ the resulting canonical form (see (65)). By assumption, there is a Jordan block with the zero eigenvalue in $\tilde{A}$, and there is a nonempty „Jordan section" (again corresponding to the zero eigenvalue) in $\tilde{B}$. Let s be the order of this section, and let $r = n - s$.

We write $\tilde{A}$ and $\tilde{B}$ as the block diagonal matrices

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ 0 & I_s \end{pmatrix}, \quad \tilde{B} = \begin{pmatrix} I_r & 0 \\ 0 & B_2 \end{pmatrix}. \tag{71}$$

The same block partition is used for the matrix $Y = P^{-1} X Q^{-1}$:

$$Y = \begin{pmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{pmatrix}. \tag{72}$$

Instead of (70), we have

$$\tilde{A}\, Y = Y^T \tilde{B}, \tag{73}$$

which implies

$$A_1 Y_{12} = Y_{21}^T B_2, \quad Y_{21} = Y_{12}^T,$$

whence

$$A_1 Y_{12} = Y_{12} B_2. \tag{74}$$

This is a homogeneous Sylvester equation. Since A_1 has a zero eigenvalue and B_2 has only the zero eigenvalue, Eq. (74) admits a nonzero solution Y_{12}. Setting

$$Y_{21} = -Y_{12}^T, \quad Y_{11} = 0, \quad Y_{22} = 0,$$

we obtain a nontrivial solution to Eq. (73).

Thus, in the subsequent analysis, we can assume that either A or B is nonsingular. Consider first the simpler case where B is nonsingular. Then, in Eq. (73),

$$\tilde{B} = I_n, \quad \tilde{A} = J_{m_1} \oplus \cdots \oplus J_{m_f} \quad (m_1 + \cdots + m_f = n).$$

The Jordan matrix $\tilde{A}$ may have blocks corresponding to the same eigenvalue. We change the order of the blocks (if required) to write $\tilde{A}$ as

$$\tilde{A} = A_1 \oplus A_2 \oplus \cdots \oplus A_k, \qquad (75)$$

where, as before, each block A_i has the only eigenvalue λ_i but the eigenvalues λ_i and λ_j corresponding to distinct blocks A_i and A_j are distinct themselves.

We partition Y in accordance with (75); thus,

$$Y = (Y_{ij})_{i,j=1}^{k}.$$

Then Eq. (73) implies

$$A_i Y_{ii} = Y_{ii}^T, \qquad i = 1, 2, \ldots, k. \quad (76)$$

From this, we derive

$$Y_{ii} = Y_{ii}^T A_i^T$$

and

$$Y_{ii}^T = A_i Y_{ii}^T A_i^T. \qquad (77)$$

This discrete Sylvester equation has only the trivial solution if $\lambda_i^2 \neq 1$ (see (57)).

Let us see what happens if $\lambda_i = \pm 1$. We drop the index i in (76) and show that the equation

$$AY = Y^T, \tag{78}$$

where A is a matrix of order $n \geq 2$ with the single eigenvalue 1 or -1, always has a nonzero solution. (If $n = 1$, then (78) reads as $\lambda y = y$; hence, $y \neq 0$ only if $\lambda = 1$.)

If $A = I_n$ (or $A = -I_n$), then every symmetric (skew-symmetric) matrix solves (78). It follows that, if A contains a subblock I_2 (resp., $-I_2$), then (78) admits a nonzero solution of the form

$$\begin{pmatrix} Y_2 & 0 \\ 0 & 0 \end{pmatrix},$$

where Y_2 is a symmetric (skew-symmetric) matrix of order 2.

Now consider the case where $A = J_n(1)$. Then, equating the entries in position (n, n) in (78), we obtain the identity $y_{nn} = y_{nn}$. This means that Eq. (78), considered as a homogeneous system of linear equations for the n^2 unknowns y_{ij}, has at most $n^2 - 1$ linearly independent equations. Such a system always has a nonzero solution. It follows that, if A contains a

Jordan block $J_k(1)$ of order $k \leq n$, then (78) admits a nonzero solution of the form

$$\begin{pmatrix} Y_k & 0 \\ 0 & 0 \end{pmatrix}, \quad Y_k \neq 0. \qquad (79)$$

Finally, we consider the case where $A = J_n(-1)$. Here, we equate in (78) the entries in positions (n, n), $(n, n-1)$, and $(n-1, n)$. This yields

$$-y_{nn} = y_{nn} \Rightarrow y_{nn} = 0,$$
$$-y_{n,n-1} = y_{n-1,n},$$
$$-y_{n-1,n} + y_{nn} = y_{n,n-1}.$$

Since $y_{nn} = 0$, the last two equations are in fact the same equation. Thus, the homogeneous system equivalent to Eq. (78) again has fewer linearly independent equations than the number n^2 of unknowns in this system; hence, there exist nontrivial solutions. If A contains a Jordan block $J_k(-1)$ of order $k \leq n$, then (78) admits a nonzero solution of form (79).

We summarize this part of our analysis as follows: For Eq. (70) to have only the trivial solution, the pair (A, B^T) should not have the eigenvalue 1 and the multiple eigenvalue -1.

Suppose that this condition is fulfilled, and assume that the pair (A, B^T) has at least two distinct finite eigenvalues λ_i and λ_j. Then, for the corresponding blocks Y_{ij} and Y_{ji}, we derive from (73) the relations

$$A_i Y_{ij} = Y_{ji}^T, \quad A_j Y_{ji} = Y_{ij}^T. \tag{80}$$

Combining them, we obtain

$$A_j Y_{ij}^T A_i^T = Y_{ij}^T.$$

According to (57), this equation has a nonzero solution Y_{ij} if $\lambda_i \lambda_j = 1$. Setting

$$Y_{ji} = Y_{ij}^T A_i^T,$$

we have

$$A_j Y_{ji} = A_j Y_{ij}^T A_i^T = Y_{ij}^T;$$

that is, both equations in (80) are satisfied. Letting

$$Y_{ii} = 0, \quad Y_{jj} = 0,$$

we see that

$$\begin{pmatrix} A_i & 0 \\ 0 & A_j \end{pmatrix} \begin{pmatrix} 0 & Y_{ij} \\ Y_{ji} & 0 \end{pmatrix} = \begin{pmatrix} 0 & Y_{ji}^T \\ Y_{ij}^T & 0 \end{pmatrix}.$$

If required, the matrix $\begin{pmatrix} 0 & Y_{ij} \\ Y_{ji} & 0 \end{pmatrix}$ can be complemented by zeros to obtain a nontrivial solution to the entire Eq. (78)

(44) From these considerations, we derive the following conclusion: For Eq. (70) to have only the trivial solution, none of the finite eigenvalues λ_i and λ_j of the pair (A, B^T) should satisfy the relation $\lambda_i \lambda_j = 1$.

Assume that both conditions concerning the finite eigenvalues are fulfilled. Then, reversing the above argument and using the unique solvability criteria for Sylvester equations, we conclude that Eq. (70) has only the trivial solution.

It remains to examine the case where B is singular; then $\det A \neq 0$. Then $\tilde{A}$ and $\tilde{B}$ can again be represented in form (71), while Y is represented as in (72). For Y_{11} to be a zero matrix, it is necessary and sufficient that the above conditions for the finite eigenvalues are fulfilled. Equation (74) has now only the trivial solution because A_1 is nonsingular, while B_2 has only the zero eigenvalue. Thus, $Y_{12} = 0$ and $Y_{21} = 0$. Finally, we have

$$Y_{22} = Y_{22}^T B_2,$$

which implies

$$Y_{22} = B_2^T \, Y_{22} \, B_2.$$

According to criterion (57), $Y_{22} = 0$ and, hence, $Y = 0$.

Thus, in the regular case, Eq. (60) is uniquely solvable for every right-hand side C if and only if the following conditions are fulfilled:

(a) At least one of the matrices A and B is nonsingular.

(b) The pair (A, B^T) has no eigenvalue 1 and no multiple eigenvalue -1.

(c) None of the pairs of distinct eigenvalues λ_i and λ_j satisfies the relation

$$\lambda_i \lambda_j = 1.$$

Solving Eq. (60) numerically

The use of the Weierstrass form in practical computations is not a good idea because the reduction to this form is as unstable numerically as the Jordan reduction. Thus, we need a different form of a matrix pair that can be attained by unitary equivalence transformations. Fortunately, such a form does exist and is an analog of the matrix Schur form:

(46) Let (E, F) be a regular pair of $n \times n$ matrices. Then there exist unitary matrices $U, V \in M_n$ such that

$$R = UEV \quad \text{and} \quad S = UFV \qquad (81)$$

are (upper) triangular matrices. The pair (R, S) is called a generalized Schur form of the pair (E, F).

For medium-size matrices E and F, there exists a numerically stable implementation of the generalized Schur reduction called the QZ algorithm. You can find a description of this algorithm in the chapter on the unsymmetric eigenvalue problem in Golub & Van Loan's book "Matrix Computations".

Let us see how the QZ algorithm can be applied to the numerical solution of Eq. (60). We assume that this equation satisfies the above conditions for unique solvability.

As was the case with the Sylvester equations, the procedure for the numerical solution of Eq. (60) consists of four stages.

(a) Reduction of A and B^T to the generalized Schur form.

In accordance with (65) and (81), we find unitary matrices U and V such that

$$R = UAV \quad \text{and} \quad S^T = UB^TV \qquad (82)$$

are upper triangular matrices. This is done by using the QZ algorithm.

(b) Transformation of the right-hand side:

$$D = UCU^T. \qquad (83)$$

As a result of Stages (a) and (b), the original Eq. (60) is replaced by a new equation of the same type; namely,

$$RY - Y^TS' = D, \qquad Y = V^*XU^T. \qquad (84)$$

(c) Solving Eq. (84).

The left relation in (84) is a block triangular system of linear equations in a **disguised** form. Let us demonstrate this for $n = 3$. For definiteness, we choose the row-wise order for both y_{ij} and d_{ij}. Then the matrix of system (84) has the form shown at the next page. The 9th equation determines y_{33}. The difference $r_{33} - s_{33}$ cannot be zero; otherwise, the pair (R, S^T) and, hence, the original pair (A, B^T) has 1 as an eigenvalue (show this!). Now that we know y_{33}, the 6th and 8th equations are a system for y_{23} and y_{32}. The determinant Δ of this system is

$$\begin{bmatrix}
r_{11}-s_{11} & 0 & 0 & r_{12}-s_{21} & 0 & 0 & r_{13}-s_{31} & 0 & 0 \\
0 & r_{11} & 0 & -s_{22} & r_{12} & 0 & -s_{32} & r_{13} & 0 \\
0 & 0 & r_{11} & 0 & 0 & r_{12}-s_{33} & 0 & r_{13} & \\
0 & -s_{11} & 0 & r_{22} & -s_{21} & 0 & r_{23} & -s_{31} & 0 \\
0 & 0 & 0 & 0 & r_{22}-s_{22} & 0 & 0 & r_{23}-s_{32} & 0 \\
0 & 0 & 0 & 0 & 0 & r_{22} & 0 & -s_{33} & r_{23} \\
0 & 0 & -s_{11} & 0 & 0 & -s_{21} & r_{33} & 0 & -s_{31} \\
0 & 0 & 0 & 0 & 0 & -s_{22} & 0 & r_{33} & -s_{32} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & r_{33}-s_{33}
\end{bmatrix}$$

equal to $r_{22} r_{33} - s_{22} s_{33}$. If at least one of the entries s_{22} and s_{33} is zero, then S and B are singular matrices; hence, A and R are nonsingular, and

$$\Delta = r_{22} r_{33} \neq 0$$

If both s_{22} and s_{33} are nonzero, then

$$\Delta = s_{22} s_{33} (\lambda_2 \lambda_3 - 1),$$

where λ_2 and λ_3 are eigenvalues of the pair (A, B^T). Thus, we again have $\Delta \neq 0$ because of the conditions imposed on the eigenvalues.

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The cost of Stage (a) is $\mathcal{O}(m^3) + \mathcal{O}(n^3)$ flops, while each of the three other stages costs $\mathcal{O}(m^2 n) + \mathcal{O}(mn^2)$ flops.

$$\begin{bmatrix} 0 \\ 0 \\ r_{13} \\ 0 \\ 0 \\ r_{23} \\ -s_{31} \\ -s_{32} \\ r_{33} - s_{33} \end{bmatrix}$$

ie

singular

r). Thus,

imposed

The next step is to consider the 3rd and 7th equations as a system for y_{13} and y_{31}. The determinant of this system is $r_{11} r_{33} - s_{11} s_{33}$ and is again nonzero. Now the 5th equation uniquely determines y_{22}. Then, from the 2nd and 4th equations, we find y_{12} and y_{21}, and the remaining first equation determines y_{11}.

(d) Returning to the original matrix X.

Inverting the second relation in (84), we obtain

$$X = V Y \bar{U}.$$