The continuous Sylvester equation

This is an equation of the form

$$AX - XB = C, \qquad (1)$$

where A and B are given square matrices of possibly different orders m and n, while a given matrix (matrice) C and the unknown matrix X are of size $m \times n$. In what follows, all the matrices are considered over $\mathbb{C}$. The set of $m \times n$ - matrices is denoted by $M_{m,n}$. If $m = n$, then we simply write M_n.

Let us replace (1) by an equivalent system of scalar equalities obtained by equating the mn entries of the left-hand and right-hand sides. Now we see that (1) is nothing else than a system of mn linear equations (sistema di equazioni lineari) for the mn unknowns x_{ij}. We can give (1) the more conventional form

$$\mathcal{A}x = b, \qquad (2)$$

using the concept of the Kronecker product (also called direct product or tensor product) of matrices. The Kronecker product of $A \in M_{m,n}$ and $B \in M_{p,q}$

is the block matrix

$$A \otimes B = \begin{pmatrix} a_{11} B & \dots & a_{1n} B \\ \dots & \dots & \dots \\ a_{m1} B & \dots & a_{mn} B \end{pmatrix} \in M_{mp, nq}.$$

Here are some useful properties of Kronecker products:

(1) $(\alpha A) \otimes B = A \otimes (\alpha B) = \alpha (A \otimes B) \quad \forall \alpha \in \mathbb{C}$

(2) $(A + B) \otimes C = A \otimes C + B \otimes C$

(3) $A \otimes (B + C) = A \otimes B + A \otimes C$

(4) Assume that the conventional products AB and CD make sense. Then

$$AB \otimes CD = (A \otimes C)(B \otimes D)$$

Problem 1. Find the inverse of the matrix
(5.5.36)

$$\mathcal{A} = \begin{bmatrix} 1 & -2 & 2 & -4 \\ -2 & 3 & -4 & 6 \\ 3 & -6 & 5 & -10 \\ -6 & 9 & -10 & 15 \end{bmatrix}.$$

Hint: Use the properties of Kronecker products.

Solution: Define

$$B = \begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}, \qquad A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$$

Then $\quad \mathcal{A} = A \otimes B.$

Both A and B are nonsingular matrices, and

$$A^{-1} = \begin{pmatrix} -5 & 2 \\ 3 & -1 \end{pmatrix}, \qquad B^{-1} = \begin{pmatrix} -3 & -2 \\ -2 & -1 \end{pmatrix}$$

From property (4), we can easily deduce that $\mathcal{A}$ is a nonsingular matrix and

$$\mathcal{A}^{-1} = A^{-1} \otimes B^{-1}.$$

Therefore,

$$\begin{bmatrix} 15 & 10 & -6 & -4 \\ 10 & 5 & -4 & -2 \\ -9 & -6 & 3 & 2 \\ -6 & -3 & 2 & 1 \end{bmatrix}$$

$\blacksquare$

Now we return to the Sylvester equation (1). We even consider the more general equation

$$A_1 X B_1 + A_2 X B_2 + \dots + A_k X B_k = C. \qquad (3)$$

Eq. (1) is a particular case of (3) that corresponds to $k = 2$, $\quad A_1 = A$, $B_1 = I_n$, $\quad A_2 = -I_m$, $B_2 = B$.

We want to rewrite (3) as a linear system of form (3). The exact form of this system depends on the orderings that we impose on the entries of X

and on the scalar equalities between the entries of the left-hand and right-hand sides in (3). For definiteness assume that, in both cases, we choose the row-wise order. According to this order, X can be considered as the following vector of dimension mn:

$$x = \operatorname{vec} X = \begin{bmatrix} x_{11} \\ x_{12} \\ \cdots \\ x_{1n} \\ x_{21} \\ x_{22} \\ \cdots \\ x_{2n} \\ \cdots \\ x_{m1} \\ x_{m2} \\ \cdots \\ x_{mn} \end{bmatrix}.$$

The vector $c = \operatorname{vec} C$ is defined analogously.

Now the linear system equivalent to Eq. (3) is

$$A \operatorname{vec} X = \operatorname{vec} C, \qquad (4)$$

where

$$\mathcal{A} = A_1 \otimes B_1^T + A_2 \otimes B_2^T + \dots + A_k \otimes B_k^T. \qquad (5)$$

In the particular case of the Sylvester equation (1), we obtain

$$\mathcal{A} = A \otimes I_n - I_m \otimes B^T. \qquad (6)$$

Matrix (6) is sometimes called the Kronecker difference of A and B.

System (4) is inconsistent, uniquely solvable, or has an infinite set of solutions depending on the solution set of the corresponding homogeneous system

$$\mathcal{A} \operatorname{vec} X = 0_{mn},$$

which, for Eq. (1), corresponds to the matrix equation

$$AX = XB. \qquad (7)$$

Therefore, we first examine Eq. (7).

In principle, an analysis of this equation can be performed using the properties of Kronecker products. For instance, one can first show the following:

<u>Problem 2.</u> Let $\lambda_i(A)$ $(i = 1, 2, \dots, m)$ and $\lambda_j(B)$ autovalori $(j = 1, 2, \dots, n)$ be the eigenvalues of A and B, respec-

tively. Then the mn scalars

$$\lambda_i(A) - \lambda_j(B), \quad i = 1,2,\dots,m;\ j = 1,2,\dots,n,$$

constitute the spectrum of matrix (6).

An immediate implication of this fact is the following assertion:

<u>Problem 3.</u> Equation (1) is uniquely solvable for every right-hand side C if and only if

$$\lambda_i(A) \neq \lambda_j(B) \quad \forall i,j; \qquad (8)$$

that is, if A and B have disjoint spectra.

However, we examine Eq. (7) using a different approach based on the Jordan canonical (or normal) form. Recall that the Jordan form of $A \in M_n$ is a matrix of the simplest possible form in the similarity class of A, which is the set

$$\mathcal{S}_A = \{ P^{-1} A P \mid P \in M_n,\ \det P \neq 0 \}.$$

For almost every A, the Jordan form is the diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_n)$ formed of the eigenvalues $\lambda_1, \dots, \lambda_n$ of A. This is certainly true if A has disjoint eigenvalues. However, this is also true for many matrices with multiple eigenvalues.

For instance, the Jordan form of every Hermitian (and, more generally, every normal) matrix is a diagonal matrix.

In the most general case, the Jordan form of A is a block diagonal matrix in which every diagonal block of order $k \geq 2$ is the so-called Jordan block

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \cdot & \cdot & \cdot \\ & & & \lambda & 1 \\ & & & & \lambda \end{bmatrix} = \lambda I_k + H_k . \qquad (9)$$

Now we return to Eq. (7). Assume that

$$A = U J_A U^{-1}, \qquad B = V J_B V^{-1}, \qquad (10)$$

where J_A and J_B are the Jordan forms of A and B:

$$J_A = J_{p_1}(\lambda_1) \oplus J_{p_2}(\lambda_2) \oplus \cdots \oplus J_{p_u}(\lambda_u) \qquad (11)$$

$$(p_1 + p_2 + \cdots + p_u = m),$$

$$J_B = J_{q_1}(\mu_1) \oplus J_{q_2}(\mu_2) \oplus \cdots \oplus J_{q_v}(\mu_v) \qquad (12)$$

$$(q_1 + q_2 + \cdots + q_v = n).$$

It is possible that $\lambda_i = \lambda_j$ ($\mu_k = \mu_l$) for $i \neq j$ ($k \neq l$).

Upon substitution of (10) into (7), we have

$$U J_A U^{-1} X = X V J_B V^{-1}$$

or

$$J_A \; U^{-1} X V = U^{-1} X V J_B . \qquad (13)$$

Define

$$Y = U^{-1} X V. \qquad (14)$$

Then (13) takes the same form

$$J_A Y = Y J_B \qquad (15)$$

as the original Eq. (7). However, the Jordan matrices J_A and J_B look much simpler than A and B.

Now we partition Y into blocks in accordance with J_A and J_B:

$$Y = (Y_{\alpha\beta}), \qquad \alpha = 1, 2, \dots, u; \; \beta = 1, 2, \dots, v. \quad (16)$$

Then Eq. (15) splits into uv matrix equations of the form

$$J_{p_\alpha}(\lambda_\alpha) \, Y_{\alpha\beta} = Y_{\alpha\beta} \, J_{q_\beta}(\mu_\beta). \qquad (17)$$

These equations are independent of each other. To simplify the notation, we drop the indices and consider the equation

$$J_k(\lambda) \, Y = Y \, J_e(\mu). \qquad (18)$$

Using representation (9), we rewrite (18) as

$$(\mu - \lambda) \, Y = H_k Y - Y H_e . \qquad (19)$$

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There are two possible cases: $\mu \neq \lambda$ and $\mu = \lambda$.

<u>Case 1</u>: $\mu \neq \lambda$. From (19), we derive

$$(\mu-\lambda)^2 Y = H_k (\mu-\lambda) Y - (\mu-\lambda) Y H_l =$$

$$H_k^2 Y - H_k Y H_l - H_k Y H_l + H_l^2 .$$

Continuing in this way or using induction, we can show that

$$(\mu-\lambda)^r Y = \sum_{s+t=r} (-1)^t \alpha_{st} H_k^s Y H_l^t , \quad r = 1, 2, 3, \ldots \qquad (20)$$

Now note that

$$H_k^m = 0, \quad m = k, k+1, \ldots$$

If, in (20),

$$r \geq k + l - 1,$$

then

$$\text{either} \quad s \geq k \quad \text{or} \quad t \geq l$$

and, hence,

$$\text{either} \quad H_k^s = 0 \quad \text{or} \quad H_l^t = 0.$$

It follows that $(\mu-\lambda)^r Y = 0$. Since $\mu \neq \lambda$, we have

$$Y = 0.$$

Thus, in Case 1, Eq. (19) has only the trivial solution.

<u>Problem 4.</u> Prove the „if" part of Problem 3.

<u>Case 2</u>: $\mu = \lambda$. Equation (19) takes the form

$$H_k Y = Y H_l. \qquad (21)$$

Writing this equality entrywise, we have

$$\begin{bmatrix} y_{21} & y_{22} & \cdots & y_{2l} \\ y_{31} & y_{32} & \cdots & y_{3l} \\ \cdot & \cdot & \cdot & \cdot \\ y_{k1} & y_{k2} & \cdots & y_{kl} \\ 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0 & y_{11} & \cdots & y_{1,l-1} \\ 0 & y_{21} & \cdots & y_{2,l-1} \\ \cdot & \cdot & \cdot & \cdot \\ 0 & y_{k-1,1} & \cdots & y_{k-1,l-1} \\ 0 & y_{k1} & \cdots & y_{k,l-1} \end{bmatrix}$$

Hence,

$$y_{21} = y_{31} = \cdots = y_{k1} = y_{k2} = \cdots = y_{k,l-1} = 0, \qquad (22)$$

$$y_{st} = y_{s-1,\,t-1}, \quad s \geq 2,\ t \geq 2. \qquad (23)$$

Consider three different subcases:

(a) $k = l$. Equalities (22), combined with (23), say that all the subdiagonal entries of Y are zero. The entries $y_{11}, y_{12}, \ldots, y_{1k}$ in the first row can be chosen arbitrarily:

$$y_{11} = c_1,\ y_{12} = c_2, \ldots, y_{1k} = c_k.$$

Then, according to (23), Y is the triangular Toeplitz matrix

$$T_k = \begin{pmatrix} c_1 & c_2 & c_3 & \cdots & c_{k-1} & c_k \\ 0 & c_1 & c_2 & \cdots & c_{k-2} & c_{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & c_1 & c_2 \\ 0 & 0 & 0 & \cdots & 0 & c_1 \end{pmatrix}$$

Thus, the solution set is in this case a k-dimensional linear subspace. ^sottospazio^

(b) $k < \ell$. Since

$$y_{k1} = y_{k2} = \cdots = y_{k,\ell-1} = 0,$$

we have

$$y_{st} = 0 \text{ if } s-t \in \{k-1, k-2, \ldots, k-\ell+1\}.$$

It follows that the first $\ell - k$ columns of Y are zero. The remaining entries are covered by the same considerations as in subcase (a). Thus, Y is a matrix of the form

$$Y = (\underbrace{0}_{\ell-k} \quad T_k)$$

The solution set is again a k-dimensional linear subspa

(c) $k > \ell$. Since

$$y_{21} = y_{31} = \cdots = y_{k1} = 0,$$

we have

$$y_{st} = 0 \quad \text{if} \quad s-t \in \{1, 2, \ldots, k-1\}.$$

It follows that the last $k-\ell$ rows of Y are zero, and Y is a matrix of the form

$$\begin{array}{r} \\ k-\ell\,\{ \end{array}\begin{pmatrix} T_\ell \\ 0 \end{pmatrix}$$

The solution set is an l-dimensional linear subspace.

Summarizing, we conclude that the solution set of Eq. (21) is a linear subspace of dimension

$$\min\{k, l\}.$$

Now we return to Eq. (15). With a pair of blocks $J_{p_\alpha}(\lambda_\alpha)$ and $J_{q_\beta}(\mu_\beta)$, we associate the value

$$\delta_{\alpha\beta} = \begin{cases} 0, & \lambda_\alpha \neq \mu_\beta, \\ \min\{p_\alpha, q_\beta\}, & \lambda_\alpha = \mu_\beta \end{cases} \tag{24}$$

Then the above analysis leads us to the following conclusion: the solution set of Eq. (15) is a linear subspace in $M_{m,n}$ whose dimension is given by the formula

$$N_{A,B} = \sum_{\alpha=1}^{u} \sum_{\beta=1}^{v} \delta_{\alpha\beta} \tag{25}$$

In view of (14), the same is the dimension of the solution set of Eq. (7).

Problem 5. Solve the matrix equation
(Ортега, с 254)

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} X - X \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{26}$$

Both A and B are triangular matrices with the eigenvalues $\lambda_1 = \lambda_2 = 1$ and $\mu_1 = \mu_2 = 2$, respectively.

According to problem 3, Eq. (26) is uniquely solvable. We write this equation as a system of linear equations. From (4) and (6), we have

$$A = \begin{pmatrix} -1 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \qquad c = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$$

Note that triangular matrices A and B give rise to a triangular system (4). Solving system $Ax = c$, we have $x_4 = x_{22} = -1$, $x_3 = x_{21} = -1 - x_4 = 0$, $x_2 = x_{12} = -1 + x_4 = -2$, $x_1 = x_{11} = -1 - x_2 + x_3 = 1$. Thus,

$$X = \begin{pmatrix} 1 & -2 \\ 0 & -1 \end{pmatrix}.$$

Problem 6. Solve the matrix equation
(Эргеза, с. 254)

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix} X + X \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} = I_2. \qquad (27)$$

Here $B = -A^T$, $\lambda_1 = \lambda_2 = -1$, $\mu_1 = \mu_2 = 1$. Thus, Eq. (27) is uniquely solvable. However, if X is a solution to (27), then X^T is also a solution. It follows that the unique solution X to (27) must be a symmetric matrix. Let us verify this by calculations. We have

$$-2x_{22} = 1 \rightarrow x_{22} = -\tfrac{1}{2}$$
$$-2x_{21} + x_{22} = 0 \rightarrow x_{21} = \frac{x_{22}}{2} = -\tfrac{1}{4}$$
$$-2x_{12} + x_{22} = 0 \rightarrow x_{12} = x_{21} = -\tfrac{1}{4}$$
$$-2x_{11} + x_{12} + x_{21} = 1 \rightarrow x_{11} = -\tfrac{3}{4}$$
$$X = -\frac{1}{4}\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

Thus, X is indeed a symmetric matrix. Moreover, X is negative definite. One could anticipate this result by recognizing a Lyapunov matrix equation in (27). The matrix A in this equation is stable (i.e., its spectrum belongs to the left half-plane), while the right-hand side is positive definite. A theorem proved by Lyapunov says that, under these conditions, the unique solution to the Lyapunov equation must be negative definite.

Some applications

A. Testing similarity

How can one verify whether or not given $n \times n$ matrices A and B are similar? The standard answer given by textbooks on linear algebra is: find the Jordan forms J_A and J_B of A and B. If J_A and J_B are the same matrix (up to the order of diagonal blocks), then A and B are similar; otherwise, they are not.

However, to construct the Jordan form of A, one must first find its eigenvalues. As Galois showed around 1830, if the ordinary operations of arithmetic and root extraction are used, there are no formulas for the zeroes of general polynomials of degree five or higher. Since any finite sequence of computer calculations corresponds to a formula of some sort, the roots of general algebraic equations cannot be found by radicals. This implies that the eigenvalues of a matrix, being the zeros of its characteristic polynomial, cannot be found by radicals.

Nevertheless, it turns out that an answer to the question of the similarity between A and B can be given as a result of a finite purely arithmetic calculation (radicals are not used!). From formula (25), one can gather that

$$N_{A,B} \le N_{A,A} \quad \text{and} \quad N_{A,B} \le N_{B,B}$$

Problem 7.* Prove that A and B are similar if and only if

$$N_{A,A} = N_{B,B} = N_{A,B} \tag{28}$$

or

$$N_{A,B}^2 = N_{A,A}\, N_{B,B}. \tag{29}$$

Hint: Consult the paper M.A. Gauger, C.I. Byrnes. Characteristic free, improved decidability criteria for the similarity problem // Linear and Multilinear Algebra 5 (3) (1977) 153 - 158.

Looking at formula (25) for $N_{A,B}$ and definition (24) of $\sigma_{\alpha\beta}$, one may feel that, to use criterion (28) or (29), one should again first find the eigenvalues of A and B. But this is not true. According to the rank-nullity theorem as applied to Kronecker difference (6),

$$\operatorname{rank} \mathcal{A} + \operatorname{nullity} \mathcal{A} = mn.$$

However, nullity $\mathcal{A}$ is the dimension of $\ker \mathcal{A}$, that is, the dimension of the null-space of the matrix operator $F(X) = AX - XB$. In other words,

$$\text{nullity } \mathcal{A} = N_{A,B}.$$

In other words, $N_{A,B}$ can be calculated using the formula

$$N_{A,B} = mn - \operatorname{rank} \mathcal{A}.$$

Moreover, if we set

$$r_{A,B} = \operatorname{rank} \mathcal{A}, \quad r_{A,A} = \operatorname{rank}(A \otimes I_m - I_m \otimes A^T),$$

$$r_{B,B} = \operatorname{rank}(B \otimes I_n - I_n \otimes B^T),$$

then (29) can be rewritten as

$$r_{A,B}^2 = r_{A,A}\, r_{B,B}. \qquad (30)$$

<u>Problem 7.</u> Using a finite computation involving only (Икрамов, 6.4.32) arithmetic operations, find out which of the matrices

$$A = \begin{pmatrix} 3 & 1 & -1 \\ -3 & -1 & 3 \\ -2 & -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 5 & -2 \\ -2 & -1 & 1 \\ -1 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 6 & 0 & 8 \\ 3 & 2 & 6 \\ -2 & 0 & -2 \end{pmatrix}$$

are similar.

Hint: Use criterion (30) and the rank procedure from your favourite computer algebra system.

Answer: A and C are similar; B is not.

<u>Problem 8.</u> Do the same for the matrices (Икрамов, 6.

(18) $A = \begin{pmatrix} -1 & -1 & 2 \\ 3 & -5 & 6 \\ 2 & -2 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -8 & 12 & -6 \\ -10 & 18 & -10 \\ -12 & 24 & -14 \end{pmatrix}$,

$$C = \begin{pmatrix} 0 & 6 & 6 \\ -2 & 16 & 12 \\ 4 & -28 & -20 \end{pmatrix}.$$

Answer: A and B are similar; C is not.

B. Commuting matrices

For a given matrix $A \in M_n$, consider the following question: how many are matrices $X \in M_n$ that commute with A, that is, satisfy the relation

$$AX = XA. \qquad (31)$$

Obviously, Eq. (31) is a particular case of (7). Hence, the solutions to (31) form a linear subspace of dimension $N_{A,A}$ in M_n.

Let J_A be the Jordan form of A as defined in (10) and (11). The summands $\delta_{\alpha\beta}$ of $N_{A,A}$ should now be defined as

$$\delta_{\alpha\beta} = \begin{cases} 0, & \lambda_\alpha \neq \lambda_\beta, \\ \min\{p_\alpha, p_\beta\}, & \lambda_\alpha = \lambda_\beta, \end{cases} \qquad (\alpha, \beta = 1, 2, \dots, u).$$

Note that

$$\delta_{\alpha\alpha} = p_\alpha, \quad \alpha = 1, 2, \ldots, u,$$

and

$$p_1 + p_2 + \cdots + p_u = n.$$

Therefore, we always have

$$N_{A,A} \geqslant n. \tag{32}$$

For $N_{A,A}$ to have the minimum possible value n, all the summands $\delta_{\alpha\beta}$ with $\alpha \neq \beta$ must be zero. This happens if and only if the eigenvalues $\lambda_1, \ldots, \lambda_u$ are pairwise distinct, in other words, if and only if there is a unique Jordan block associated with each eigenvalue. A matrix $A \in M_n$ with this property is said to be nonderogatory.

Problem 9. Nonderogatory matrices can also be described as the matrices with each eigenvalue having geometric multiplicity 1. Prove that a matrix $A \in M_n$ is nonderogatory if and only if the minimal polynomial of A is identical to its characteristic polynomial.

Problem 10. Based on problem 9, propose a finite algorithm, involving only arithmetic operations, for verifying whether a given matrix $A \in M_n$ is nonderogatory.

C. Similarity of a matrix and its transpose

From $A = UJ_A U^{-1}$, we derive

$$A^T = U^{-T} J_A^T U^T.$$

The transposed matrix J_A^T can be obtained from J_A by a symmetric rearrangement of the rows and columns (check this for a single Jordan block!), which is the simplest similarity transformation. Thus, $J_A^T \sim J_A$ and, hence, $A^T \sim A$; that is,

$$A^T = P^{-1} A P \qquad (33)$$

for some nonsingular matrix $P \in M_n$.

It is obvious that the matrix P in (33) is a solution to the matrix equation

$$AX = XA^T. \qquad (34)$$

Since A and A^T have the same Jordan form, inequality (32) is also true for N_{A,A^T}:

$$N_{A,A^T} \geq n. \qquad (35)$$

Moreover, the equality in (35) is attained if and only if A is a nonderogatory matrix.

Now we describe symmetric solutions to Eq.(34). To this end, we add the symmetry condition to (34):

$$AX = XA^T, \quad X = X^T. \tag{36}$$

Since X is symmetric, the first equality in (36) is equivalent to the requirement that AX be a symmetric matrix.

Taken together, the two equalities (36) constitute a homogeneous system of linear equations for the n^2 unknown entries x_{ij}. It consists of

$$n^2 - n = \frac{n(n-1)}{2} + \frac{n(n-1)}{2}$$

scalar symmetry conditions for the matrices X and AX. Since the rank of this system is at most n^2-n, the rank-nullity theorem says that its solution set is a linear subspace of dimension at least $n^2-(n^2-n) = n$.

Suppose that A in (34) is a nonderogatory matrix. Then, instead of (35), we have

$$N_{A,A^T} = n.$$

On the other hand, the symmetric solutions to (34) form

(22)

a subspace of dimension $\geq n$. We arrive at the following conclusion:

If $A \in M_n$ is a nonderogatory matrix, then all the solutions to Eq. (34) are symmetric matrices. In particular, each nonsingular matrix that transforms A into A^T by similarity is symmetric.

Problem 11. (Икрамов, 6.4.23) Let

$$A = \begin{pmatrix} 11 & 4 \\ -4 & 3 \end{pmatrix}.$$

Describe all the solutions to the equation $AX = XA^T$. Are they symmetric matrices? How do you explain this?

Problem 12. (Икрамов, 6.4.83) Do the same for the matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & 2 & 1 \end{bmatrix}.$$

Solving the Sylvester equation numerically

As noted earlier, Eq. (1) is a system of $N = mn$

linear equations for the N unknown entries x_{ij}. There are great many methods for solving linear systems, which can be divided into two groups depending on the type of system; namely, the matrix of a system is a stored or a large-scale one. A stored matrix is one for which all N^2 entries are stored in computer memory. The order of such a matrix is small or medium. Stored systems are usually solved by direct methods such as Gaussian elimination and its modifications. Large-scale systems are for the most part treated by iterative methods.

The methods for solving Sylvester equations are divided into two similar groups. For lack of time, we discuss only methods for a small or medium N.

A straightforward approach to solving Eq. (1) is to convert it into a system of linear equations and apply Gaussian elimination. The cost of this approach is $\mathcal{O}(N^3) = \mathcal{O}(m^3 n^3)$ arithmetic operations. We show that Eq. (1) can be solved much more efficiently by using the eigenvalue methods. However, unlike in the above theoretical analysis, we do not utilize the reduction to Jordan form, which is known to be a numerically highly unstable procedure. This reduction is usually achieved by a nonunitary similarity transformation, and a general principle, effective in numerical linear algebra, says: if the same problem can be solved using either unitary or nonunitary transformations, then, from the viewpoint of numerical stability, the unitary approach should be preferred.

There is a useful matrix decomposition attained by a unitary similarity transformation. It is called the Schur decomposition because it was first introduced in the important Schur's unitary triangularization theorem:

Given $A \in M_n$ with the eigenvalues $\lambda_1, \ldots, \lambda_n$ in any prescribed order, there is a unitary matrix $V \in M_n$ such that

$$R = V^* A V \qquad (37)$$

is an (upper) triangular matrix with the diagonal entries $r_{ii} = \lambda_i$ $(i = 1, 2, \ldots, n)$.

For medium-size matrices, there exists a numerically stable implementation of the Schur reduction called the QR algorithm. I assume that you know what the QR algorithm is from numerical analysis courses that you have taken. It is enough to recall that the calculation of the triangular matrix R and the unitary matrix U in (37) costs $O(n^3)$ flops.

In 1972, Bartels and Stewart proposed a numerical procedure, based on the QR algorithm, for solving stored Sylvester equations.

The Bartels - Stewart algorithm consists of the following stages (it is assumed that conditions (8) are fulfilled):

(a) Reduction of A and B to their Schur forms.

Along with (37), we calculate at this stage the Schur decomposition of B:

$$S = V^* B V. \tag{38}$$

(b) Transformation of the right-hand side:

$$D = U^* C V. \tag{39}$$