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Nonuniform Sampling and Reconstruction in Shift-Invariant Spaces*

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Abstract. This article discusses modern techniques for nonuniform sampling and reconstruction of functions in shift-invariant spaces. It is a survey as well as a research paper and provides a unified framework for uniform and nonuniform sampling and reconstruction in shift-invariant spaces by bringing together wavelet theory, frame theory, reproducing kernel Hilbert spaces, approximation theory, amalgam spaces, and sampling. Inspired by applications taken from communication, astronomy, and medicine, the following aspects will be emphasized: (a) The sampling problem is well defined within the setting of shift-invariant spaces. (b) The general theory works in arbitrary dimension and for a broad class of generators. (c) The reconstruction of a function from any sufficiently dense nonuniform sampling set is obtained by efficient iterative algorithms. These algorithms converge geometrically and are robust in the presence of noise. (d) To model the natural decay conditions of real signals and images, the sampling theory is developed in weighted L^p-spaces.

Key words. nonuniform sampling, irregular sampling, sampling, reconstruction, wavelets, shift-invariant spaces, frame, reproducing kernel Hilbert space, weighted L^p-spaces, amalgam spaces

AMS subject classifications. 41A15,42C15, 46A35, 46E15, 46N99, 47B37

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1. Introduction.

Modern digital data processing of functions (or signals or images) always uses a discretized version of the original signal f that is obtained by sampling f on a discrete set. The question then arises whether and how f can be recovered from its samples. Therefore, the objective of research on the sampling problem is twofold. The first goal is to quantify the conditions under which it is possible to recover particular classes of functions from different sets of discrete samples. The second goal is to use these analytical results to develop explicit reconstruction schemes for the analysis and processing of digital data. Specifically, the sampling problem consists of two main parts:

(a) Given a class of functions V on $\mathbb{R}^d$, find conditions on sampling sets $X = \{x_j \in \mathbb{R}^d : j \in J\}$, where J is a countable index set, under which a function $f \in V$ can be reconstructed uniquely and stably from its samples $\{f(x_j) : x_j \in X\}$.

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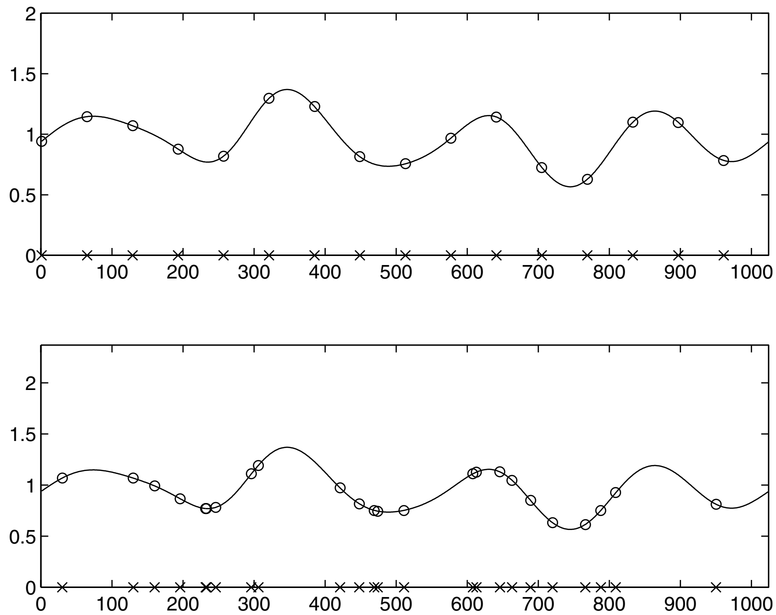

Fig. 1.1 *The sampling problem. Top: A function f defined on $\mathbb{R}$ has been sampled on a uniform grid. Bottom: The same function f has been sampled on a nonuniformly spaced set. The sampling locations x_j are marked by the symbol $\times$, and the sampled values $f(x_j)$ by a circle* o.

(b) Find efficient and fast numerical algorithms that recover any function $f \in V$ from its samples on X.

In some applications, it is justified to assume that the sampling set $X = \{x_j : j \in J\}$ is *uniform*, i.e., that X forms a regular n-dimensional Cartesian grid; see Figures 1.1 and 1.2. For example, a digital image is often acquired by sampling light intensities on a uniform grid. Data acquisition requirements and the ability to process and reconstruct the data simply and efficiently often justify this type of uniform data collection. However, in many realistic situations the data are known only on a nonuniformly spaced sampling set. This nonuniformity is a fact of life and prevents the use of the standard methods from Fourier analysis. The following examples are typical and indicate that nonuniform sampling problems are pervasive in science and engineering.

- *Communication theory*: When data from a uniformly sampled signal (function) are lost, the result is generally a sequence of nonuniform samples. This scenario is usually referred to as a *missing data* problem. Often, missing samples are due to the partial destruction of storage devices, e.g., scratches on a CD. As an illustration, in Figure 1.3 we simulate a missing data problem by randomly removing samples from a slice of a three-dimensional magnetic resonance (MR) digital image.
- *Astronomical measurements*: The measurement of star luminosity gives rise to extremely nonuniformly sampled time series. Daylight periods and adverse nighttime weather conditions prevent regular data collection (see, e.g., [111] and the references therein).
- *Medical imaging*: Computerized tomography (CT) and magnetic resonance imaging (MRI) frequently use the nonuniform polar and spiral sampling sets (see Figure 1.2 and [21, 90]).

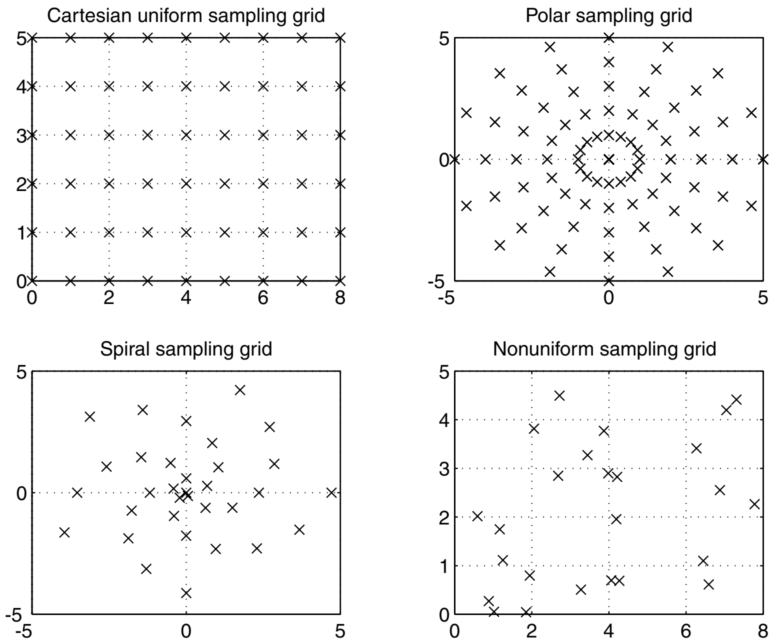

Fig. 1.2 *Sampling grids. Top left: Because of its simplicity the uniform Cartesian sampling grid is used in signal and image processing whenever possible. Top right: A polar sampling grid used in computerized tomography (see* [90]*). In this case, the two-dimensional Fourier transform $\hat{f}$ is sampled with the goal of reconstructing f. Bottom left: Spiral sampling used for fast MRI by direct signal reconstruction from spectral data on spirals* [21]*. Bottom right: A typical nonuniform sampling set as encountered in spectroscopy, astronomy, geophysics, and other signal and image processing applications.*

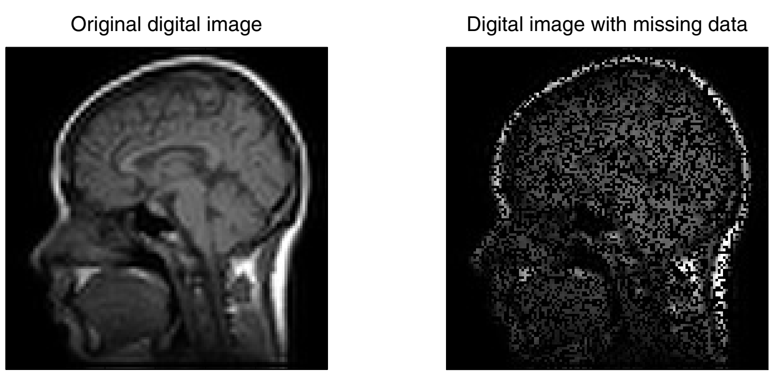

Fig. 1.3 *The missing data problem. Left: Original digital MRI image with* 128×128 *samples. Right: MRI image with* 50% *randomly missing samples.*

Other applications using nonuniform sampling sets occur in geophysics [92], spectroscopy [101], general signal/image processing [13, 22, 103, 106], and biomedical imaging [20, 59, 90, 101] (see Figures 1.2 and 1.4). More information about modern techniques for nonuniform sampling and applications can be found in [16].

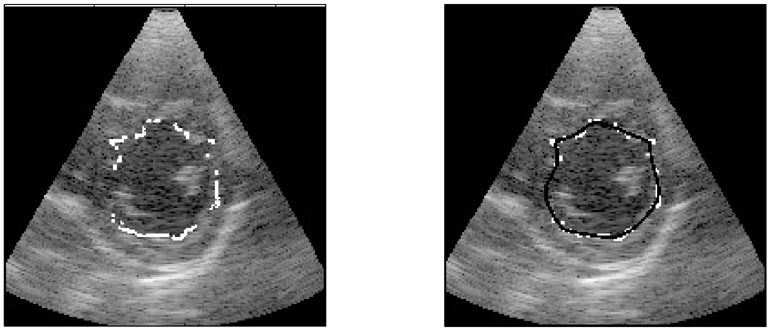

Fig. 1.4 *Sampling and boundary reconstruction from ultrasonic images. Left: Detected edge points of the left ventricle of a heart from a two-dimensional ultrasound image constitute a nonuniform sampling of the left ventricle's contour. Right: Boundary of the left ventricle reconstructed from the detected edge sample points (see* [59]*).*

1.1. Sampling in Paley–Wiener Spaces: Bandlimited Functions. Since infinitely many functions can have the same sampled values on $X = \{x_j\}_{j\in J} \subset \mathbb{R}^d$, the sampling problem becomes meaningful only after imposing some a priori conditions on f. The standard assumption is that the function f on $\mathbb{R}^d$ belongs to the space of *bandlimited* functions B_Ω; i.e., the Fourier transform $\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle \xi,x\rangle}dx$ of f is such that $\hat{f}(\xi) = 0$ for all $\xi \notin \Omega = [-\omega,\omega]^d$ for some $\omega < \infty$ (see, e.g., [15, 44, 47, 55, 62, 72, 78, 88, 51, 112] and the review papers [27, 61, 65]). The reason for this assumption is a classical result of Whittaker [114] in complex analysis which states that, for dimension $d = 1$, a function $f \in L^2(\mathbb{R}) \cap B_{[-1/2,1/2]}$ can be recovered exactly from its samples $\{f(k) : k \in \mathbb{Z}\}$ by the interpolation formula

$$f(x) = \sum_{k\in\mathbb{Z}} f(k)\,\mathrm{sinc}(x-k), \tag{1.1}$$

where $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$. This series gave rise to the uniform sampling theory of Shannon [96], which is fundamental in engineering and digital signal processing because it gives a framework for converting analog signals into sequences of numbers. These sequences can then be processed digitally and converted back to analog signals via (1.1).

Taking the Fourier transform of (1.1) and using the fact that the Fourier transform of the sinc function is the characteristic function $\chi_{[-1/2,1/2]}$ shows that for any $\xi \in [-1/2, 1/2]$

$$\hat{f}(\xi) = \sum_k f(k)e^{2\pi ik\xi} = \sum_k \langle \hat{f}, e^{i2\pi k\cdot}\rangle_{L^2(-1/2,1/2)}\, e^{i2\pi k\xi}.$$

Thus, reconstruction by means of the formula (1.1) is equivalent to the fact that the set $\{e^{i2\pi k\xi},\ k \in \mathbb{Z}\}$ forms an orthonormal basis of $L^2(-1/2, 1/2)$ called the *harmonic Fourier basis*. This equivalence between the harmonic Fourier basis and the reconstruction of a uniformly sampled bandlimited function has been extended to treat some special cases of nonuniformly sampled data. In particular, the results by Paley and Wiener [87], Kadec [71], and others on the *nonharmonic Fourier bases* $\{e^{i2\pi x_k\xi},\ k \in \mathbb{Z}\}$ can be translated into results about nonuniform sampling and reconstruction of bandlimited functions [15, 62, 89, 94]. For example, Kadec's theorem

[71] states that if $X = \{x_k \in \mathbb{R} : \ |x_k - k| \le L < 1/4\}$ for all $k \in \mathbb{Z}$, then the set $\{e^{i2\pi x_k \xi},\ k \in \mathbb{Z}\}$ is a *Riesz basis* of $L^2(-1/2, 1/2)$; i.e., $\{e^{i2\pi x_k \xi},\ k \in \mathbb{Z}\}$ is the image of an orthonormal basis of $L^2(-1/2, 1/2)$ under a bounded and invertible operator from $L^2(-1/2, 1/2)$ onto $L^2(-1/2, 1/2)$. Using Fourier transform methods, this result implies that any bandlimited function $f \in L^2 \cap B_{[-1/2,1/2]}$ can be completely recovered from its samples $f(x_k), k \in \mathbb{Z}$, as long as the sampling set is of the form $X = \{x_k \in \mathbb{R} : \ |x_k - k| < 1/4\}_{k\in\mathbb{Z}}$.

The sampling set $X = \{x_k \in \mathbb{R} : \ |x_k - k| < 1/4\}_{k\in\mathbb{Z}}$ in Kadec's theorem is just a perturbation of $\mathbb{Z}$. For more general sampling sets, the work of Beurling [23, 24], Landau [74], and others [18, 58] provides a deep understanding of the one-dimensional theory of nonuniform sampling of bandlimited functions. Specifically, for the exact and stable reconstruction of a bandlimited function f from its samples $\{f(x_j) : x_j \in X\}$, it is sufficient that the *Beurling density*

$$D(X) = \lim_{r\to\infty} \inf_{y\in\mathbb{R}} \frac{\#X \cap (y + [0, r])}{r} \tag{1.2}$$

satisfies $D(X) > 1$. Conversely, if f is uniquely and stably determined by its samples on $X \subset \mathbb{R}$, then $D(X) \ge 1$ [74]. The marginal case $D(X) = 1$ is very complicated and is treated in [79, 89, 94].

It should be emphasized that these results deal with *stable reconstructions*. This means that an inequality of the form

$$\|f\|_p \le C \left(\sum_{x_j \in X} |f(x_j)|^p \right)^{1/p}$$

holds for all bandlimited functions $f \in L^p \cap B_\Omega$. A sampling set for which the reconstruction is stable in this sense is called a *(stable) set of sampling*. This terminology is used to contrast a set of sampling with the weaker notion of a *set of uniqueness*. X is a set of uniqueness for B_Ω if $f|_X = 0$ implies that $f = 0$. Whereas a set of sampling for $B_{[-1/2,1/2]}$ has a density $D \ge 1$, there are sets of uniqueness with arbitrarily small density. See [73, 25] for examples and characterizations of sets of uniqueness.

While the theorems of Paley and Wiener and Kadec about Riesz bases consisting of complex exponentials $e^{i2\pi x_k \xi}$ are equivalent to statements about sampling sets that are perturbations of $\mathbb{Z}$, the results about arbitrary sets of sampling are connected to the more general notion of frames introduced by Duffin and Schaeffer [40]. The concept of frames generalizes the notion of orthogonal bases and Riesz bases in Hilbert spaces and of unconditional bases in some Banach spaces [2, 5, 6, 12, 14, 15, 20, 28, 29, 46, 66, 97].

1.2. Sampling in Shift-Invariant Spaces. The series (1.1) shows that the space of bandlimited functions $B_{[-1/2,1/2]}$ is identical with the space

$$V^2(\text{sinc}) = \left\{ \sum_{k\in\mathbb{Z}} c_k \,\text{sinc}(x - k) : \ (c_k) \in \ell^2 \right\}. \tag{1.3}$$

Since the sinc function has infinite support and slow decay, the space of bandlimited functions is often unsuitable for numerical implementations. For instance, the pointwise evaluation

$$f \mapsto f(x_0) = \sum_{k\in\mathbb{Z}} c_k \,\text{sinc}(x_0 - k)$$

is a nonlocal operation, because, as a consequence of the long-range behavior of sinc, many coefficients c_k will contribute to the value $f(x_0)$. In fact, all bandlimited functions have infinite support since they are analytic. Moreover, functions that are measured in applications tend to have frequency components that decay for higher frequencies, but these functions are not bandlimited in the strict sense. Thus, it has been advantageous to use non-bandlimited models that retain some of the simplicity and structure of bandlimited models but are more amenable to numerical implementation and are more flexible for approximating real data [13, 63, 64, 86, 103, 104]. One such example are the *shift-invariant spaces* which form the focus of this paper.

A shift-invariant space is a space of functions on $\mathbb{R}^d$ of the form

$$V(\phi_1, \dots, \phi_r) = \left\{ \sum_{i=1}^{r} \sum_{j \in \mathbb{Z}^d} c_j^i \phi_i(x - j) \right\}.$$

Such spaces have been used in finite elements and approximation theory [34, 35, 67, 68, 69, 98] and for the construction of multiresolution approximations and wavelets [32, 33, 39, 53, 60, 70, 82, 83, 95, 98, 99, 100]. They have been extensively studied in recent years (see, for instance, [6, 19, 52, 67, 68, 69]).

Sampling in shift-invariant spaces that are not bandlimited is a suitable and realistic model for many applications, e.g., for taking into account real acquisition and reconstruction devices, for modeling signals with smoother spectrum than is the case with bandlimited functions, or for numerical implementation [9, 13, 22, 26, 85, 86, 103, 104, 107, 110, 115, 116]. These requirements can often be met by choosing "appropriate" functions ϕ_i. This may mean that the functions ϕ_i have a shape corresponding to a particular "impulse response" of a device, or that they are compactly supported, or that they have a Fourier transform $|\hat{\phi}_i(\xi)|$ that decays smoothly to zero as $|\xi| \to \infty$.

1.2.1. Uniform Sampling in Shift-Invariant Spaces. Early results on sampling in shift-invariant spaces concentrated on the problem of uniform sampling [7, 9, 10, 11, 37, 64, 105, 108, 107, 113, 116] or interlaced uniform sampling [110]. The problem of uniform sampling in shift-invariant spaces shares some similarities with Shannon's sampling theorem in that it requires only the Poisson summation formula and a few facts about Riesz bases [7, 9]. The connection between interpolation in spline spaces, filtering of signals, and Shannon's sampling theory was established in [11, 109]. These results imply that Shannon's sampling theory can be viewed as a limiting case of polynomial spline interpolation when the order of the spline tends to infinity [11, 109]. Furthermore, Shannon's sampling theory is a special case of interpolation in shift-invariant spaces [7, 9, 113, 116] and a limiting case for the interpolation in certain families of shift-invariant spaces $V(\phi^n)$ that are obtained by a generator $\phi^n = \phi * \cdots * \phi$ consisting of the n-fold convolution of a single generator ϕ [9].

In applications, signals do not in general belong to a prescribed shift-invariant space. Thus, when using the bandlimited theory, the common practice in engineering is to force the function f to become bandlimited before sampling. Mathematically, this corresponds to multiplication of the Fourier transform $\hat{f}$ of f by a characteristic function χ_Ω. The new function f_a with Fourier transform $\hat{f}_a = \hat{f}\chi_\Omega$ is then sampled and stored digitally for later processing or reconstruction. The multiplication by χ_Ω before sampling is called *prefiltering with an ideal filter* and is used to reduce the errors in reconstructions called *aliasing errors*. It has been shown that the three steps of the traditional uniform sampling procedure, namely prefiltering, sampling, and postfiltering for reconstruction, are equivalent to finding the best L^2-approximation of a function in $L^2 \cap B_\Omega$ [9, 105]. This procedure generalizes to sampling in general shift-invariant spaces [7, 9, 10, 85, 105, 108]. In fact, the reconstruction from the samples of a function should be considered as an approximation in the shift-invariant space generated by the impulse response of the sampling device. This allows a reconstruction that optimally fits the available samples and can be done using fast algorithms [106, 107].

1.2.2. Nonuniform Sampling in Shift-Invariant Spaces. The problem of nonuniform sampling in general shift-invariant spaces is more recent [4, 5, 30, 66, 75, 76, 77, 102, 119]. The earliest results [31, 77] concentrate on perturbation of regular sampling in shift-invariant spaces and are therefore similar in spirit to Kadec's result for bandlimited functions. For the L^2 case in dimension $d = 1$, and under some restrictions on the shift-invariant spaces, several theorems on nonuniform sampling can be found in [76, 102]. Moreover, a lower bound on the maximal distance between two sampling points needed for reconstructing a function from its samples was given for the case of polynomial splines and other special cases of shift-invariant spaces in [76]. For the general multivariate case in L^p, the theory was developed in [4], and for the case of polynomial spline shift-invariant spaces, the maximal allowable gap between samples was obtained in [5]. For general shift-invariant spaces, a Beurling density $D \geq 1$ is necessary for stable reconstruction [5]. As in the case of bandlimited functions, the theory of frames is central in nonuniform sampling of shift-invariant spaces, and there is an equivalence between a certain type of frame and the problem of sampling in shift-invariant spaces [5, 66, 75].

The aim of the remainder of this paper is to provide a unified framework for uniform and nonuniform sampling in shift-invariant spaces. This is accomplished by bringing together wavelet theory, frame theory, reproducing kernel Hilbert spaces, approximation theory, amalgam spaces, and sampling. This combination simplifies some parts of the literature on sampling. We also hope that this unified theory will provide the ground for more interactions between mathematicians, engineers, and other scientists who are using the theory of sampling and reconstruction in specific applications.

The paper is intended as a survey, but it contains several new results. In particular, all the well-known results are developed in weighted L^p-spaces. Extensions of frame theory and reproducing kernel Hilbert spaces to Banach spaces are discussed, and the connections between reproducing kernels in weighted L^p-spaces, Banach frames, and sampling are described. In the spirit of a review, we focus on the discussion of the sampling problem and results, and we postpone the technical details and proofs to the end of each section or to section 8. The reader more interested in the applications and techniques can omit the proofs in a first reading.

The paper is organized as follows. Section 2 introduces the relevant spaces for sampling theory and presents some of their properties. Weighted L^p-spaces and sequence spaces are defined in section 2.1. Wiener amalgam spaces are discussed in section 2.2, where we also derive some convolution relations in the style of Young's inequalities. The weighted L^p-shift-invariant spaces are introduced in section 2.3, and some of their main properties are established. The sampling problem in weighted shift-invariant spaces is stated in section 3. In sections 4.1 and 4.2 some aspects of reproducing kernel Hilbert spaces and frame theory are reviewed. The discussion includes an extension of frame theory and reproducing kernel Hilbert spaces to Banach spaces. The connections between reproducing kernels in weighted L^p-spaces, Banach frames, and sampling are discussed in section 4.3. Frame algorithms for the reconstruction of a function from its samples are discussed in section 5. Section 6 discusses iterative reconstructions. In applications, a function f does not belong to a particular prescribed space V, in general. Moreover, even if the assumption that a function f belongs to a particular space V is valid, the samples of f are not exact due to digital inaccuracy, or the samples are corrupted by noise when they are obtained by a real measuring device. For this reason, section 7 discusses the results of the various reconstruction algorithms when the samples are corrupted by noise, which is an important issue in practical applications. The proofs of the lemmas and theorems of sections 6 and 7 are given in section 8.

2. Function Spaces.

This section provides the basic framework for treating nonuniform sampling in weighted shift-invariant spaces. The shift-invariant spaces under consideration are of the form

$$V(\phi) = \left\{ \sum_{k\in\mathbb{Z}^d} c_k \phi(\cdot - k) \right\}, \tag{2.1}$$

where $c = (c_k)_{k\in\mathbb{Z}}$ is taken from some sequence space and ϕ is the so-called *generator* of $V(\phi)$. Before it is possible to give a precise definition of shift-invariant spaces, we need to study the convergence properties of the series $\sum_{k\in\mathbb{Z}^d} c_k\phi(\cdot - k)$. In the context of the sampling problem the functions in $V(\phi)$ must also be continuous. In addition, we want to control the growth or decay at infinity of the functions in $V(\phi)$. Thus the generator ϕ and the associated sequence space cannot be chosen arbitrarily. To settle these questions, we first discuss weighted L^p-spaces with specific classes of weight functions (section 2.1), and we then develop the main properties of amalgam spaces (section 2.2). Only then will we give a rigorous definition of a shift-invariant space and derive its main properties in section 2.3. Shift-invariant spaces figure prominently in other areas of applied mathematics, notably in wavelet theory and approximation theory [33, 34]. Our presentation will be adapted to the requirements of sampling theory.

2.1. Weighted L^p_ν-Spaces.

To model decay or growth of functions, we use weighted L^p-spaces [41]. A function f belongs to $L^p_\nu(\mathbb{R}^d)$ with weight function ν if νf belongs to $L^p(\mathbb{R}^d)$. Equipped with the norm $\|f\|_{L^p_\nu} = \|\nu f\|_{L^p}$, the space L^p_ν is a Banach space. If the weight function ν grows rapidly as $|x| \to \infty$, then the functions in L^p_ν decay roughly at a corresponding rate. Conversely, if the weight function ν decays rapidly, then the functions in L^p_ν may grow as $|x| \to \infty$.

In general, a weight function is just a nonnegative function ν. We will use two special types of weight functions. The weight functions denoted by ω are always assumed to be continuous, symmetric, i.e., $\omega(x) = \omega(-x)$, positive, and submultiplicative:

$$0 < \omega(x + y) \le \omega(x)\omega(y) \quad \forall x, y \in \mathbb{R}^d. \tag{2.2}$$

This submultiplicativity condition implies that $1 \le \omega(0) \le \omega(x)$ for all $x \in \mathbb{R}^d$. For a technical reason, we impose the growth condition

$$\sum_{n=1}^{\infty} \frac{\log \omega(nk)}{n^2} < \infty \qquad \forall k \in \mathbb{Z}^d\,.$$

Although most of the results do not require this extra condition on ω, we use it in Lemma 2.11. For simplicity we refer to ω as a *submultiplicative* weight. A prototypical example is the Sobolev weight $\omega(x) = (1+|x|)^\alpha$, with $\alpha \geq 0$. When $\omega = 1$, we obtain the usual L^p-spaces.

In addition, a weight function ν is called *moderate* with respect to the submultiplicative weight ω, or simply ω-*moderate*, if it is continuous, symmetric, and positive and satisfies $\nu(x+y) \leq C\omega(x)\nu(y)$ for all $x, y \in \mathbb{R}^d$. For instance, the weights $\nu(x) = (1+|x|)^\beta$ are moderate with respect to $\omega(x) = (1+|x|)^\alpha$ if and only if $|\beta| \leq \alpha$. If ν is ω-moderate, then $\nu(y) = \nu(x+y-x) \leq C\omega(-x)\nu(x+y)$, and it follows that

$$\frac{1}{\nu(x+y)} \leq C\omega(x)\frac{1}{\nu(y)}\,.$$

Thus, the weight $\frac{1}{\nu}$ is also ω-moderate.

If ν is ω-moderate, then a simple computation shows that

$$\|f(\cdot - y)\|_{L^p_\nu} \leq C\omega(y)\,\|f\|_{L^p_\nu}\,,$$

and in particular, $\|f(\cdot - y)\|_{L^p_\omega} \leq \omega(y)\,\|f\|_{L^p_\omega}$. Conversely, if L^p_ν is translation-invariant, then $\omega(x) = \sup_{\|f\|_{L^p_\nu}\leq 1} \|f(. - x)\|_{L^p_\nu}$ is submultiplicative and ν is ω-moderate. To see this, we note that

$$\omega(x) = \sup_{\|f\|_{L^p_\nu}\leq 1} \|f(. - x)\|_{L^p_\nu}$$

is the operator norm of the translation operator $f \mapsto f(\cdot - x)$. Since operator norms are submultiplicative, it follows that $\omega(x+y) \leq \omega(x)\omega(y)$. Moreover,

$$\int_{\mathbb{R}^d} |f(t-x)|^p\,\nu(t)^p dt = \int_{\mathbb{R}^d} |f(t)|^p\,\nu(t+x)^p dt \leq \omega(x)^p \int_{\mathbb{R}^d} |f(t)|^p\,\nu(t)^p dt.$$

Thus, $\nu(x+y) \leq \omega(x)\nu(y)$, and therefore the weighted L^p-spaces with a moderate weight are exactly the translation-invariant spaces.

We also consider the weighted sequence spaces $\ell^p_\nu(\mathbb{Z}^d)$ with weight ν: a sequence $\{(c_k) :\; k \in \mathbb{Z}^d\}$ belongs to ℓ^p_ν if $((c\nu)_k) = (c_k\nu_k)$ belongs to ℓ^p with norm $\|c\|_{\ell^p_\nu} = \|\nu c\|_{\ell^p}$, where (ν_k) is the restriction of ν to $\mathbb{Z}^d$.

2.2. Wiener Amalgam Spaces. For the sampling problem we also need to control the local behavior of functions so that the sampling operation $f \mapsto (f(x_j))_{j\in J}$ is at least well defined. This is done conveniently with the help of the Wiener amalgam spaces $W(L^p_\nu)$. These consist of functions that are "locally in L^∞ and globally in L^p_ν." To be precise, a measurable function f belongs to $W(L^p_\nu)$, $1 \leq p < \infty$, if it satisfies

$$\|f\|^p_{W(L^p_\nu)} = \sum_{k\in\mathbb{Z}^d} \operatorname{ess\,sup}\{|f(x+k)|^p\,\nu(k)^p; x \in [0,1]^d\} < \infty. \tag{2.3}$$

If $p = \infty$, a measurable function f belongs to $W(L^\infty_\nu)$ if it satisfies

$$\|f\|_{W(L^\infty_\nu)} = \sup_{k\in\mathbb{Z}^d}\{\operatorname{ess\,sup}\{|f(x+k)|\,\nu(k); x \in [0,1]^d\}\} < \infty. \tag{2.4}$$

Note that $W(L^\infty_\nu)$ coincides with L^∞_ν.

Endowed with this norm, $W(L^p_\nu)$ becomes a Banach space [43, 45]. Moreover, it is translation invariant; i.e., if $f \in W(L^p_\nu)$, then $f(\cdot - y) \in W(L^p_\nu)$ and

$$\|f(\cdot - y)\|_{W(L^p_\nu)} \leq C\omega(y)\,\|f\|_{W(L^p_\nu)}\,.$$

The subspace of continuous functions $W_0(L^p_\nu) = W(C, L^p_\nu) \subset W(L^p_\nu)$ is a closed subspace of $W(L^p_\nu)$ and thus also a Banach space [43, 45]. We have the following inclusions between the various spaces.

THEOREM 2.1. *Let ν be ω-moderate and $1 \le p \le q \le \infty$. Then the following inclusions hold:*

(i) $W_0(L^p_\nu) \subset W_0(L^q_\nu)$ *and* $W(L^p_\nu) \subset W(L^q_\nu) \subset L^q_\nu$.

(ii) $W_0(L^p_\omega) \subset W_0(L^p_\nu)$, $W(L^p_\omega) \subset W(L^p_\nu)$, *and* $L^p_\omega \subset L^p_\nu$.

The following convolution relations in the style of Young's theorem [118] are useful.

THEOREM 2.2. *Let ν be ω-moderate.*

(i) *If $f \in L^p_\nu$ and $g \in L^1_\omega$, then $f * g \in L^p_\nu$ and*

$$\|f * g\|_{L^p_\nu} \le C \|f\|_{L^p_\nu} \|g\|_{L^1_\omega} .$$

(ii) *If $f \in L^p_\nu$ and $g \in W(L^1_\omega)$, then $f * g \in W(L^p_\nu)$ and*

$$\|f * g\|_{W(L^p_\nu)} \le C \|f\|_{L^p_\nu} \|g\|_{W(L^1_\omega)} .$$

(iii) *If $c \in \ell^p_\nu$ and $d \in \ell^1_\omega$, then $c * d \in \ell^p_\nu$ and*

$$\|c * d\|_{\ell^p_\nu} \le C \|c\|_{\ell^p_\nu} \|d\|_{\ell^1_\omega} .$$

REMARK 2.1. *Amalgam spaces and their generalizations have been investigated by Feichtinger, and the results of Theorem* 2.1 *can be found in* [42, 43, 45, 44]. *The results and methods developed by Feichtinger can also be used to deduce Theorem* 2.2. *However, for the sake of completeness, in section* 2.4 *we present direct proofs of Theorems* 2.1 *and* 2.2 *that do not rely on the deep results of amalgam spaces.*

2.3. Shift-Invariant Spaces. This section discusses shift-invariant spaces and their basic properties. Although some of the following observations are known in wavelet and approximation theory, they have received little attention in connection with sampling.

Given a so-called generator ϕ, we consider shift-invariant spaces of the form

$$V^p_\nu(\phi) = \left\{ \sum_{k \in \mathbb{Z}^d} c_k \phi(\cdot - k) \; : \; c \in \ell^p_\nu \right\} . \tag{2.5}$$

If $\nu = 1$, we simply write $V^p(\phi)$. The weight function ν controls the decay or growth rate of the functions in $V^p_\nu(\phi)$. To some extent, the parameter p also controls the growth of the functions in $V^p_\nu(\phi)$, but more importantly, p controls the norm we wish to use for measuring the size of our functions. For some applications in image processing, the choice $p = 1$ is appropriate [36]; $p = 2$ corresponds to the energy norm, and $p = \infty$ is used as a measure in some quality control applications. Moreover, the smoothness of a function and its appropriate value of p, $1 \le p < \infty$, for a given class of signals or images can be estimated using wavelet decomposition techniques [36]. The determination of p and the signal smoothness are used for optimal compression and coding of signals and images.

For the spaces $V^p_\nu(\phi)$ to be well defined, some additional conditions on the generator ϕ must be imposed. For $\nu = 1$ and $p = 2$, the standard condition in wavelet theory is often stated in the Fourier domain as

$$0 < m \le \hat{a}_\phi(\xi) = \sum_{j \in \mathbb{Z}^d} |\hat{\phi}(\xi + j)|^2 \le M < \infty \quad \text{for almost every } \xi, \tag{2.6}$$

for some constants $m > 0$ and $M > 0$ [80, 81]. This condition implies that $V^2(\phi)$ is a closed subspace of L^2 and that $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ is a Riesz basis of $V^2(\phi)$, i.e., the image of an orthonormal basis under an invertible linear transformation [33].

The theory of Riesz bases asserts the existence of a dual basis. Specifically, for any Riesz basis for $V^2(\phi)$ of the form $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$, there exists a unique function $\widetilde{\phi} \in V^2(\phi)$ such that $\{\widetilde{\phi}(\cdot - k) : k \in \mathbb{Z}^d\}$ is also a Riesz basis for $V^2(\phi)$ and such that $\widetilde{\phi}$ satisfies the biorthogonality relation

$$\langle \widetilde{\phi}(\cdot), \phi(\cdot - k)\rangle = \delta(k),$$

where $\delta(0) = 1$ and $\delta(k) = 0$ for $k \neq 0$. Since the dual generator $\widetilde{\phi}$ belongs to $V^2(\phi)$, it can be expressed in the form

$$\widetilde{\phi}(\cdot) = \sum_{k\in\mathbb{Z}^d} b_k \phi(\cdot - k)\,. \tag{2.7}$$

The coefficients b_k are determined explicitly by the Fourier series

$$\sum_{k\in\mathbb{Z}^d} b_k e^{2\pi i k\xi} = \left(\sum_{k\in\mathbb{Z}^d} \left|\widehat{\phi}(\xi + k)\right|^2\right)^{-1};$$

i.e., (b_k) is the inverse Fourier transform of $(\sum_{k\in\mathbb{Z}^d} |\widehat{\phi}(\xi + k)|^2)^{-1}$ (see, for example, [8, 9]). Since $a_\phi(\xi)^{-1} \leq 1/m$ by (2.6), the sequence (b_k) exists and belongs to $\ell^2(\mathbb{Z}^d)$.

In order to handle general shift-invariant spaces $V^p_\nu(\phi)$ instead of $V^2(\phi)$, we need more information about the dual generator. The following result is one of the central results in this paper and is essential for the treatment of general shift-invariant spaces.

THEOREM 2.3. *Assume that* (1) $\phi \in W(L^1_\omega)$ *and that* (2) $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ *is a Riesz basis for* $V^2(\phi)$. *Then the dual generator* $\widetilde{\phi}$ *is in* $W(L^1_\omega)$.

As a corollary, we obtain the following theorem.

THEOREM 2.4. *Assume that* $\phi \in W(L^1_\omega)$ *and* ν *is* ω*-moderate.*

(i) *The space* $V^p_\nu(\phi)$ *is a subspace (not necessarily closed) of* L^p_ν *and* $W(L^p_\nu)$ *for any* p *with* $1 \leq p \leq \infty$.

(ii) *If* $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ *is a Riesz basis of* $V^2(\phi)$, *then there exist constants* $m_p > 0, M_p > 0$ *such that*

$$m_p \|c\|_{\ell^p_\nu} \leq \left\|\sum_{k\in\mathbb{Z}^d} c_k \phi(\cdot - k)\right\|_{L^p_\nu} \leq M_p \|c\|_{\ell^p_\nu} \quad \forall c \in \ell^p_\nu(\mathbb{Z}^d) \tag{2.8}$$

is satisfied for all $1 \leq p \leq \infty$ *and all* ω*-moderate weights* ν. *Consequently,* $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ *is an unconditional basis for* $V^p_\nu(\phi)$ *for* $1 \leq p < \infty$, *and* $V^p_\nu(\phi)$ *is a closed subspace of* L^p_ν *and* $W(L^p_\nu)$ *for* $1 \leq p \leq \infty$.

The theorem says that the inclusion in Theorem 2.4(i) and the norm equivalence (2.8) hold simultaneously for all p and all ω-moderate weights, provided that they hold for the Hilbert space $V^2(\phi)$. But in $V^2(\phi)$, the Riesz basis property (2.8) is much easier to check. In fact, it is equivalent to inequalities (2.6). Inequalities (2.8) imply that ℓ^p_ν and $V^p_\nu(\phi)$ are isomorphic Banach spaces and that the set $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ is an unconditional basis of $V^p_\nu(\phi)$. In approximation theory we say that ϕ has stable integer translates and is a stable generator [67, 68, 69]. When $\nu = 1$ the conclusion (2.8) of Theorem 2.4 is well known and can be found in [67, 68, 69].

As a corollary of Theorem 2.4, we obtain the following inclusions among shift-invariant spaces.

COROLLARY 2.5. *Assume that $\phi \in W(L^1_\omega)$ and that ν is ω-moderate. Then*

$$V^1_\omega(\phi) \subset V^p_\omega(\phi) \subset V^q_\omega(\phi) \quad \textit{for } 1 \le p \le q \le \infty$$

and

$$V^q_\omega(\phi) \subset V^q_\nu(\phi) \quad \textit{for } 1 \le q \le \infty.$$

2.4. Proof of Theorems. We begin with the following properties of weight functions.

LEMMA 2.6. *Let $\mathcal{K}$ be a compact subset of $\mathbb{R}^d$ and let ν be an ω-moderate weight. Then there exists a constant $C_1 > 0$ such that*

$$C_1^{-1}\nu(j) \le \nu(x+j) \le C_1 \nu(j) \quad \forall j \in \mathbb{Z}^d,\ \forall x \in \mathcal{K}.$$

Proof. Using the submultiplicative property, we have

$$\nu(x+j) \le C\omega(x)\nu(j)$$

and

$$\nu(j) = \nu(x+j-x) \le C\nu(x+j)\omega(-x)\,.$$

We may take $C_1 = C \max_{x\in\mathcal{K}} \omega(x)$, since ω is continuous and symmetric and $\mathcal{K}$ is compact. ∎

As a consequence of Lemma 2.6 and the definition of $W(L^p_\nu)$ we obtain a slightly different characterization of the amalgam spaces.

COROLLARY 2.7. *The following are equivalent.*

(i) $f \in W(L^p_\nu)$.

(ii) $|f| \le \sum_{k\in\mathbb{Z}^d} c_k \chi_{[0,1]^d}(\cdot - k)$ *a.e. for some $c \in \ell^p_\nu$, for instance,* $c_k = \operatorname{ess\,sup}_{x\in[0,1]^d} |f(x+k)|$.

In the corollary above, we used the standard notation $\chi_{[0,1]^d}$ to denote the characteristic function of $[0,1]^d$.

Proof (of Theorem 2.1). Write $b_l = \operatorname{ess\,sup}_{x\in[0,1]^d} |f(x+l)\nu(l)|$. Then $\|b\|_{\ell^p} = \|f\|_{W(L^p_\nu)}$ for $1 \le p \le \infty$. Therefore, the inclusions $W_0(L^p_\nu) \subset W_0(L^q_\nu)$ and $W(L^p_\nu) \subset W(L^q_\nu)$ in (i) follow immediately from the inclusion $\ell^p \subset \ell^q$ when $1 \le p \le q \le \infty$. The inclusion $W_0(L^p_\nu) \subset W(L^p_\nu)$ is obvious.

Next, using Lemma 2.6, we deduce that

$$\begin{aligned}\int_{\mathbb{R}^d} |f(x)\nu(x)|^p\,dx &= \int_{[0,1]^d} \sum_{j\in\mathbb{Z}^d} |f(x+j)\nu(x+j)|^p\,dx \\ &\le C_1 \int_{[0,1]^d} \sum_{j\in\mathbb{Z}^d} |f(x+j)\nu(j)|^p\,dx \le C_1\,\|f\|^p_{W(L^p_\nu)}\end{aligned} \tag{2.9}$$

holds for $1 \le p < \infty$. Consequently, the inclusion $W(L^p_\nu) \subset L^p_\nu$ holds.

Similarly, for $p = \infty$ we have

$$\begin{aligned}\|f\nu\|_{L^\infty} &= \sup_{j\in\mathbb{Z}^d}\{\operatorname{ess\,sup}\{|f(x+j)\nu(x+j)| :\ x\in[0,1]^d\}\} \\ &\le C_1 \sup_{j\in\mathbb{Z}^d}\{\operatorname{ess\,sup}\{|f(x+j)\nu(j)| :\ x\in[0,1]^d\}\} \\ &= C_1\,\|f\|_{W(L^\infty_\nu)}.\end{aligned} \tag{2.10}$$

The inclusion $L^p_\omega \subset L^p_\nu$ follows immediately from the inequality $\nu(x) \le C\nu(0)\omega(x)$ for all $x \in \mathbb{R}^d$. Likewise, the inclusion $W(L^p_\omega) \subset W(L^p_\nu)$ follows from $\ell^p_\omega \subset \ell^p_\nu$. □

Proof (of Theorem 2.2). To prove (i), let $f \in L^p_\nu$, $g \in L^1_\omega$, and $1 \le p \le \infty$. Then using the fact that $\nu(x) = \nu(x-y+y) \le C\nu(x-y)\omega(y)$, we have

$$\begin{aligned} |(f*g)(x)|\,\nu(x) &= \left|\int_{\mathbb{R}^d} g(y)f(x-y)dy\right|\nu(x) \\ &\le C\int_{\mathbb{R}^d} |g(y)|\,\omega(y)\,|f(x-y)|\,\nu(x-y)dy \\ &\le C(|g|\,\omega * |f|\,\nu)(x). \end{aligned} \tag{2.11}$$

From the pointwise estimate above and Young's inequality for the convolution of an L^1 function with an L^p function, it follows that

$$\|(f*g)\nu\|_{L^p} \le C\,\|f\nu\|_{L^p}\,\|g\omega\|_{L^1}.$$

Thus $f*g \in L^p_\nu$ and $\|f*g\|_{L^p_\nu} \le C\,\|f\|_{L^p_\nu}\,\|g\|_{L^1_\omega}$.

To prove (ii), consider first the case $g = \chi_{[0,1]^d}$ for $1 \le p < \infty$. Write $b_k = \operatorname{ess\,sup}_{x\in[0,1]^d} |f*\chi_{[0,1]^d}(x+k)|$. Then, using Hölder's inequality, we obtain

$$b_k^p \le \operatorname{ess\,sup}_{x\in[0,1]^d}\left|\int_{[0,1]^d} |f(x+k-y)|\,dy\right|^p \le \int_{[0,1]^d-[0,1]^d} |f(k-y)|^p\,dy.$$

Using Lemma 2.6 with $\mathcal{K} = [0,1]^d - [0,1]^d = [-1,1]^d$, it follows that

$$\begin{aligned} \|b\|^p_{\ell^p_\nu} &\le \int_{[0,1]^d-[0,1]^d} \sum_{k\in\mathbb{Z}^d} |f(k-y)|^p\,|\nu(k)|^p dy \\ &\le C_1 \textstyle\int_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} |f(k-y)|^p\,|\nu(k-y)|^p dy = C_1\,\|f\|^p_{L^p_\nu}. \end{aligned} \tag{2.12}$$

Thus we have $\left\|f*\chi_{[0,1]^d}\right\|_{W(L^p_\nu)} \le C\,\|f\|_{L^p_\nu}$.

For general $g \in W(L^1_\omega)$ we use the representation of Corollary 2.7, which implies that $|g| \le \sum_{k\in\mathbb{Z}^d} c_k\chi_{[0,1]^d}(\cdot - k)$ and $\|c\|_{\ell^1_\omega} = \|g\|_{W(L^1_\omega)}$. We estimate

$$|f*g| \le |f|*|g| \le \sum_{k\in\mathbb{Z}^d} c_k\left(|f|*\chi_{[0,1]^d}\right)(\cdot - k),$$

and consequently

$$\begin{aligned} \|f*g\|_{W(L^p_\nu)} &\le \sum_{k\in\mathbb{Z}^d} c_k\left\||f|*\chi_{[0,1]^d}(\cdot - k)\right\|_{W(L^p_\nu)} \\ &\le C_2\sum_{k\in\mathbb{Z}^d} c_k\,\omega(k)\,\|f\|_{L^p_\nu}. \end{aligned}$$

The last inequality implies

$$\|f*g\|_{W(L^p_\nu)} \le C_2\,\|f\|_{L^p_\nu}\,\|g\|_{W(L^1_\omega)}.$$

The case $p = \infty$ is proved in a similar fashion.

The proof of (iii) is similar to the proof of (i). □

To finish the proofs of this section, we need the following three lemmas.

Lemma 2.8. *If $\phi \in W(L^1_\omega)$ then the autocorrelation sequence*

$$a_k = \int_{\mathbb{R}^d} \phi(x)\overline{\phi(x-k)}dx \tag{2.13}$$

belongs to ℓ^1_ω, and we have

$$\|a\|_{\ell^1_\omega} \leq C\,\|\phi\|^2_{W(L^1_\omega)}\,.$$

Proof. Write $b_k = \text{ess sup}_{x\in[0,1]^d}|\phi(x+k)|$ and $b^\vee_k = b_{-k} = \text{ess sup}_{x\in[0,1]^d}|\phi(x-k)|$. Then $\|\phi\|_{W(L^1_\omega)} = \|b\|_{\ell^1_\omega} = \|b^\vee\|_{\ell^1_\omega}$ and

$$\begin{aligned}|a_k| &\leq \int_{\mathbb{R}^d}|\phi(x)|\,|\phi(x-k)|\,dx\\ &\leq \int_{[0,1]^d}\sum_{j\in\mathbb{Z}^d}|\phi(x+j)|\,|\phi(x+j-k)|dx \leq \sum_{j\in\mathbb{Z}^d} b_j b_{j-k}\\ &= (b * b^\vee)(k).\end{aligned}$$

Theorem 2.2(iii) implies that $\|a\|_{\ell^1_\omega} \leq C\|b\|^2_{\ell^1_\omega} = C\,\|\phi\|^2_{W(L^1_\omega)}$ □

Lemma 2.9. *If $\phi \in W(L^1_\omega)$ and $c \in \ell^p_\nu$, then the function $f = \sum_{k\in\mathbb{Z}^d} c_k\phi(x-k)$ belongs to $W(L^p_\nu)$ and*

$$\|f\|_{W(L^p_\nu)} \leq C\,\|c\|_{\ell^p_\nu}\,\|\phi\|_{W(L^1_\omega)}\,.$$

Proof. Write $b_k = \text{ess sup}_{x\in[0,1]^d}|\phi(x+k)|$, $d_k = \text{ess sup}_{x\in[0,1]^d}|f(x+k)|$. Then $\|\phi\|_{W(L^1_\omega)} = \|b\|_{\ell^1_\omega}$ and $\|f\|_{W(L^p_\nu)} = \|d\|_{\ell^p_\nu}$, and we have

$$d_k = \text{ess sup}_{x\in[0,1]^d}\left|\sum_{j\in\mathbb{Z}^d} c_j\phi(x+k-j)\right| \leq \sum_{j\in\mathbb{Z}^d}|c_j|\,b_{k-j} = (|c| * b)(k).$$

Theorem 2.2(iii) then implies that $\|d\|_{\ell^p_\nu} \leq C\,\|c\|_{\ell^p_\nu}\,\|b\|_{\ell^1_\omega}$; in other words, $\|f\|_{W(L^p_\nu)} \leq C\|c\|_{\ell^p_\nu}\,\|\phi\|_{W(L^1_\omega)}$. □

Lemma 2.10. *If $f \in L^p_\nu$ and $g \in W(L^1_\omega)$, then the sequence d defined by $d_k = \int_{\mathbb{R}^d} f(x)\overline{g(x-k)}dx$ belongs to ℓ^p_ν and we have*

$$\|d\|_{\ell^p_\nu} \leq C\,\|f\|_{L^p_\nu}\,\|g\|_{W(L^1_\nu)}\,, \qquad 1 \leq p \leq \infty\,.$$

Remark 2.2. *The fact that the autocorrelation sequence in Lemma* 2.8 *belongs to ℓ^1_ω is a direct consequence of Lemma* 2.10.

Proof. Since $g \in W(L^1_\omega) \subset L^{p'}_{1/\nu}$ by Theorem 2.1 and $f \in L^p_\nu$, the terms d_k are well defined. For $1 \leq p < \infty$ we have

$$\begin{aligned}|d_k\nu(k)|^p &= \left|\int_{\mathbb{R}^d} f(x)\overline{g(x-k)}\nu(k)dx\right|^p\\ &\leq \left(\int_{[0,1]^d}\sum_{j\in\mathbb{Z}^d}|f(x+j)|\,|g(x+j-k)\nu(k)|dx\right)^p\\ &\leq \int_{[0,1]^d}\left(\sum_{j\in\mathbb{Z}^d}|f(x+j)|\,|g(x+j-k)\nu(k)|\right)^p dx.\end{aligned}$$

We sum over k and apply Theorem 2.2(iii) to the sequences $\{f(x+j) : j \in \mathbb{Z}^d\}$ and $\{g(x-j) : j \in \mathbb{Z}^d\}$ for fixed $x \in \mathbb{R}^d$, and we obtain

$$\begin{aligned}\|d\|_{\ell^p_\nu}^p &\le \int_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} \left| \sum_{j\in\mathbb{Z}^d} |f(x+j)|\,|g(x+j-k)\nu(k)| \right|^p dx \\ &\le C^p \int_{[0,1]^d} \sum_{k\in\mathbb{Z}^d} |f(x+k)\nu(k)|^p \left(\sum_{k\in\mathbb{Z}^d} |g(x-k)\omega(k)| \right)^p dx \\ &\le C^p \|g\|^p_{W(L^1_\omega)} \|f\|^p_{L^p_\nu} .\end{aligned}$$

The case $p = \infty$ is proved in a similar fashion. □

For the proof of Theorem 2.3 we need the following weighted version of Wiener's lemma on absolutely convergent Fourier series.

LEMMA 2.11. *Assume that the submultiplicative weight ω satisfies the so-called Beurling–Domar condition (mentioned in section* 2.1*)*

$$\sum_{n=1}^{\infty} \frac{\log \omega(nk)}{n^2} < \infty \quad \forall\, k \in \mathbb{Z}^d . \tag{2.14}$$

If $f(\xi) = \sum_{k\in\mathbb{Z}^d} a_k e^{2\pi i k\xi}$ is an absolutely convergent Fourier series with coefficient sequence $a = (a_k)_{k\in\mathbb{Z}^d} \in \ell^1_\omega(\mathbb{Z}^d)$ and if $f(\xi) \neq 0$ for all $\xi \in \mathbb{R}^d$, then $\frac{1}{f}$ also has an absolutely convergent Fourier series $\frac{1}{f(\xi)} = \sum_{k\in\mathbb{Z}^d} b_k e^{2\pi i k\xi}$ with coefficient sequence $b = (b_k)_{k\in\mathbb{Z}^d} \in \ell^1_\omega(\mathbb{Z}^d)$.

REMARK 2.3. *The unweighted version is a classical lemma of Wiener. The weighted version is implicit in* [38] *and stated in* [91]*.*

We are now ready to prove Theorem 2.3.

Proof (of Theorem 2.3). We have already seen that the dual generator $\widetilde{\phi} \in V^2(\phi)$ has the expansion

$$\widetilde{\phi} = \sum_{k\in\mathbb{Z}^d} b_k \phi(\cdot - k) ,$$

where the coefficients b_k are the Fourier coefficients of $\hat{a}^{-1}(\xi) = (\sum_{k\in\mathbb{Z}^d} |\hat{\phi}(\xi+k)|^2)^{-1}$. We wish to apply Lemma 2.11 to $\hat{a}$. Since $\{\phi(\cdot - k) : k \in \mathbb{Z}^d\}$ is a Riesz basis for $V^2(\phi)$, we have $\hat{a}(\xi) \neq 0$ for all $\xi \in \mathbb{R}^d$ by (2.6). Furthermore, using the Poisson summation formula, $\hat{a}$ has the Fourier series

$$\hat{a}(\xi) = \sum_{k\in\mathbb{Z}^d} |\hat{\phi}(\xi+k)|^2 = \sum_{k\in\mathbb{Z}^d} \langle \phi, \phi(\cdot - k)\rangle e^{2\pi i k\xi} .$$

Consequently, by Lemma 2.8, the Fourier coefficients of $\hat{a}$ are in $\ell^1_\omega(\mathbb{Z}^d)$. Thus the hypotheses of Wiener's lemma are satisfied, and we conclude that the Fourier coefficients of $\hat{a}^{-1}$ are also in $\ell^1_\omega(\mathbb{Z}^d)$. Now Lemma 2.9 implies that $\widetilde{\phi} \in W(L^1_\omega)$. □

Proof (of Theorem 2.4). Part (i) and the right-hand inequality in (2.8) follow directly from Lemma 2.9.

To prove the remaining statements, we consider the operator T_ϕ defined by

$$\mathrm{T}_\phi\, c = \sum_{k\in\mathbb{Z}^d} c_k \phi(\cdot - k) , \quad c \in \ell^p_\nu, \tag{2.15}$$

and the operator $\mathrm{T}^*_{\widetilde{\phi}}$ defined by

$$(\mathrm{T}^*_{\widetilde{\phi}} f)_k = \int_{\mathbb{R}^d} f(x)\overline{\widetilde{\phi}(x-k)}dx\,. \tag{2.16}$$

Lemma 2.9 implies that T_ϕ is a bounded map from ℓ^p_ν to L^p_ν with range $V^p_\nu(\phi)$. Furthermore, Lemma 2.10 implies that $\mathrm{T}^*_{\widetilde{\phi}}$ is a bounded map from L^p_ν to ℓ^p_ν.

Let $f = \sum_{k\in\mathbb{Z}^d} c_k\phi(\cdot-k) \in V^p_\nu(\phi) = \mathrm{Range}(\mathrm{T}_\phi)$. Since $\{\widetilde{\phi}(\cdot-k) : k\in\mathbb{Z}^d\}$ is biorthogonal to $\{\phi(\cdot-k) : k\in\mathbb{Z}^d\}$, we find that $c_k = \langle f, \widetilde{\phi}(\cdot-k)\rangle = (\mathrm{T}^*_{\widetilde{\phi}} f)_k$, or $c = \mathrm{T}^*_{\widetilde{\phi}} f$. Consequently,

$$\|c\|_{\ell^p_\nu} \le \|\mathrm{T}^*_{\widetilde{\phi}}\|_{\mathrm{op}}\,\|f\|_{L^p_\nu}\,, \tag{2.17}$$

and we may choose $m_p = \|\mathrm{T}^*_{\widetilde{\phi}}\|^{-1}_{\mathrm{op}}$ as the lower bound in (2.8). The other statements of the theorem follow immediately from (2.8). □

Proof (of Corollary 2.5). Since $\nu(k) = \nu(k+0) \le C\nu(0)\omega(k)$, we immediately have the inclusions $\ell^q_\omega(\mathbb{Z}^d) \subset \ell^q_\nu(\mathbb{Z}^d)$. Since

$$\ell^1_\omega(\mathbb{Z}^d) \subset \ell^p_\omega(\mathbb{Z}^d) \subset \ell^q_\omega(\mathbb{Z}^d) \subset \ell^q_\nu \quad \text{for } 1\le p\le q\le\infty,$$

the inequality (2.8) then implies the inclusions

$$V^1_\omega(\phi) \subset V^p_\omega(\phi) \subset V^q_\omega(\phi) \subset V^q_\nu(\phi) \quad \text{for } 1\le p\le q\le\infty. \qquad \square$$

3. The Sampling Problem in Weighted Shift-Invariant Spaces $V^p_\nu(\phi)$. For a reasonable formulation of the sampling problem in $V^p_\nu(\phi)$ the point evaluations $f \to f(x)$ must be well defined. Furthermore, a small variation in the sampling point should produce only a small variation in the sampling value. As a minimal requirement, we need the functions in $V^p_\nu(\phi)$ to be continuous. This is guaranteed by the following statement.

THEOREM 3.1. *Assume that $\phi \in W_0(L^1_\omega)$, that ϕ satisfies* (2.6), *and that ν is ω-moderate.*

(i) $V^p_\nu(\phi) \subset W_0(L^p_\nu)$ *for all* $p, 1\le p\le\infty$.

(ii) *If $f \in V^p_\nu(\phi)$, then we have the norm equivalences*

$$\|f\|_{L^p_\nu} \approx \|c\|_{\ell^p_\nu} \approx \|f\|_{W(L^p_\nu)}\,.$$

(iii) *If $X = \{x_j : j\in J\}$ is such that $\inf_{j,l}|x_j - x_l| > 0$, then*

$$\left(\sum_{x_k\in X} |f(x_k)|^p\,|\nu(x_k)|^p\right)^{1/p} \le C_p\,\|f\|_{L^p_\nu} \quad \forall\, f\in V^p_\nu(\phi). \tag{3.1}$$

In particular, if ϕ is continuous and has compact support, then the conclusions (i)–(iii) *hold.*

A set $X = \{x_j : j\in J\}$ satisfying $\inf_{j,l}|x_j - x_l| > 0$ is called *separated*.

Inequality (3.1) has two interpretations. It implies that the *sampling operator* $\mathrm{S_X} : f \to f|_X$ is a bounded operator from $V^p_\nu(\phi)$ into the corresponding sequence space

$$\ell^p_\nu(X) = \left\{(c_j) : \left(\sum_{j\in J}|c_j|^p\nu(x_j)^p\right)^{1/p} < \infty\right\}.$$

Equivalently, the *weighted sampling operator* $S_X : f \to f\nu|_X$ is a bounded operator from $V^p_\nu(\phi)$ into ℓ^p.

To recover a function $f \in V^p_\nu(\phi)$ from its samples, we need a converse of inequality (3.1). Following Landau [74], we say that X is a *set of sampling* for $V^p_\nu(\phi)$ if

$$c_p \|f\|_{L^p_\nu} \le \left(\sum_{x_j \in X} |f(x_j)|^p \, |\nu(x_j)|^p \right)^{1/p} \le C_p \|f\|_{L^p_\nu} , \tag{3.2}$$

where c_p and C_p are positive constants independent of f.

The left-hand inequality implies that if $f(x_j) = 0$ for all $x_j \in X$, then $f = 0$. Thus X is a set of uniqueness. Moreover, the sampling operator S_X can be inverted on its range and $S_X{}^{-1}$ is a bounded operator from Range(S_X) $\subset \ell^p_\nu(X)$ to $V^p_\nu(\phi)$. Thus (3.2) says that a small change of a sampled value $f(x_j)$ causes only a small change of f. This implies that the sampling is stable or, equivalently, that the reconstruction of f from its samples is continuous. As pointed out in section 1.1, every set of sampling is a set of uniqueness, but the converse is not true. For practical considerations and numerical implementations, only sets of sampling are of interest, because only these can lead to robust algorithms.

A solution to the sampling problem consists of two parts:

(a) Given a generator ϕ, we need to find conditions on X, usually in the form of a density, such that the norm equivalence (3.2) holds. Then, at least in principle, $f \in V^p_\nu(\phi)$ is uniquely and stably determined by $f|_X$.

(b) We need to design reconstruction procedures that are useful and efficient in practical applications. The objective is to find efficient and fast numerical algorithms that recover f from its samples $f|_X$, when (3.2) is satisfied.

Remark 3.1.

(i) *The hypothesis that X be separated is for convenience only and is not essential. For arbitrary sampling sets, we can use* adaptive weights *to compensate for the local variations of the sampling density* [48, 49]. *Let $V_j = \{x \in \mathbb{R}^d : |x - x_j| \le |x - x_k| \text{ for all } k \ne j\}$ be the Voronoi region at x_j, and let $\gamma_j = \lambda(V_j)$ be the size of V_j. Then X is a set of sampling for $V^p_\nu(\phi)$ if*

$$c_p \|f\|_{L^p_\nu} \le \left(\sum_{x_j \in X} |f(x_j)|^p \, \gamma_j \, |\nu(x_j)|^p \right)^{1/p} \le C_p \|f\|_{L^p_\nu} .$$

In numerical applications the adaptive weights γ_j are used as a cheap device for preconditioning and for improving the ratio C_p/c_p, the condition number of the set of sampling [49, 101].

(ii) *The assumption that the samples $\{f(x_j) : j \in J\}$ can be measured exactly is not realistic. A better assumption is that the sampled data is of the form*

$$g_{x_j} = \int_{\mathbb{R}^d} f(x)\psi_{x_j}(x)dx, \tag{3.3}$$

where $\{\psi_{x_j} : x_j \in X\}$ is a set of functionals that act on the function f to produce the data $\{g_{x_j} : x_j \in X\}$. The functionals $\{\psi_{x_j} : x_j \in X\}$ may reflect the characteristics of the sampling devices. For this case, the well-posedness *condition* (3.2) *must be replaced by*

$$c_p \|f\|_{L^p} \le \left(\sum_{x_j \in X} \left| g_{x_j}(f) \right|^p \right)^{1/p} \le C_p \|f\|_{L^p} , \tag{3.4}$$

where g_{x_j} *are defined by* (3.3) *and where* c_p *and* C_p *are positive constants independent of* f [1].

3.1. Proof of Theorem 3.1.

Proof. To prove (i), let $f = \sum_{k\in\mathbb{Z}^d} c_k \phi(\cdot - k) \in V^p_\nu(\phi)$. Then Lemma 2.9 implies that

$$\|f\|_{W(L^p_\nu)} \le C \|c\|_{\ell^p_\nu} \|\phi\|_{W(L^1_\omega)} . \tag{3.5}$$

To verify the continuity of f in the case $1 \le p < \infty$, we observe that $W(L^p_\nu) \subset W(L^\infty_\nu) \subset L^\infty_\nu$ and thus

$$\|f\|_{L^\infty_\nu} \le C \|f\|_{W(L^p_\nu)} . \tag{3.6}$$

Let $f_n = \nu(\cdot) \sum_{|k|\le n} c_k \phi(\cdot - k)$ be a partial sum of f. Then (3.5) and (3.6) imply that

$$\|f - f_n\|_{L^\infty_\nu} \le C \|\phi\|_{W(L^1_\omega)} \left(\sum_{|k|>n} |c_k|^p \nu(k)^p \right)^{1/p} .$$

Therefore, the sequence of continuous functions νf_n converges uniformly to the continuous function νf. Since ν is positive and continuous, f must be continuous as well.

To treat the case $p = \infty$ we choose a sequence ϕ_n of continuous functions with compact support such that $\|\phi - \phi_n\|_{W(L^1_\omega)} \mapsto 0$ as $n \to \infty$. Set $f_n(x) = \sum_{k\in Z} c_k \phi_n(x - k)$. Since the sum is locally finite, each f_n is continuous. Using (3.5) we estimate

$$\|f - f_n\|_{L^\infty_\nu} \le C \|c\|_{\ell^\infty_\nu} \|\phi - \phi_n\|_{W(L^1_\omega)} \to 0.$$

It follows that the sequence $f_n \nu$ converges uniformly to $f\nu$. Thus f is continuous as well.

Regarding the proof of (ii), the norm equivalence $\|f\|_{L^p_\nu} \approx \|c\|_{\ell^p_\nu}$ was proved earlier in Theorem 2.4. Theorem 2.1 implies that $\|f\|_{L^p_\nu} \le C \|f\|_{W(L^p_\nu)}$. Finally, if $f = \sum_k c_k \phi(\cdot - k) \in V^p_\nu(\phi)$, then we obtain

$$\|f\|_{W(L^p_\nu)} \le C \|c\|_{\ell^p_\nu} \|\phi\|_{W(L^1_\omega)} \le C_1 \|f\|_{L^p_\nu}$$

by Lemma 2.9 and (2.8). This proves that $\|f\|_{L^p_\nu}$ and $\|f\|_{W(L^p_\nu)}$ are equivalent norms on $V^p_\nu(\phi)$.

For the proof (iii), if $\inf_{j,l} |x_j - x_l| = \delta > 0$, then there are at most $N = N(\delta)$ sampling points in every cube $k + [0,1]^d$. Thus, using Lemma 2.6, we obtain

$$\begin{aligned} \sum_{x_j \in k+[0,1]^d} |f(x_j)|^p |\nu(x_j)|^p &\le N \sup_{x\in[0,1]^d} |f(x+k)|^p |\nu(x+k)|^p \\ &\le CN \sup_{x\in[0,1]^d} |f(x+k)|^p |\nu(k)|^p . \end{aligned}$$

Taking the sum over $k \in \mathbb{Z}^d$ and applying the norm equivalence proved in (ii), we obtain

$$\begin{aligned}\sum_{x_j \in X} |f(x_j)|^p \, |\nu(x_j)|^p &\le CN \sum_{k\in\mathbb{Z}^d} \sup_{x\in[0,1]^d} |f(x+k)|^p \, |\nu(k)|^p \\ &= NC_1 \, \|f\|^p_{W(L^p_\nu)} \\ &\le C_2 \, \|f\|_{L^p_\nu}\end{aligned}$$

for all $f \in V^p_\nu(\phi)$. ◻

4. Reproducing Kernel Hilbert Spaces, Frames, and Nonuniform Sampling.

As mentioned in the introduction, results of Paley and Wiener and Kadec relate Riesz bases consisting of complex exponentials to sampling sets that are perturbations of $\mathbb{Z}$. More generally, the appropriate concept for arbitrary sets of sampling in shift-invariant spaces is the concept of frames discussed in section 4.2. Frame theory generalizes and encompasses the theory of Riesz bases and enables us to translate the sampling problem into a problem of functional analysis. The connection between frames and sets of sampling is established by means of reproducing kernel Hilbert spaces (RKHSs), discussed in the next section. Frames are introduced in section 4.2, and the relation between RKHSs, frames, and sets of sampling is developed in section 4.3.

4.1. RKHSs. Theorem 3.1(iii) holds for arbitrary separated sampling sets, so in particular Theorem 3.1(iii) shows that all point evaluations $f \to f(x)$ are continuous linear functionals on $V^p_\nu(\phi)$ for all $x \in \mathbb{R}^d$. Since $V^p_\nu(\phi) \subset L^p_\nu$ and the dual space of L^p_ν is $L^{p'}_{1/\nu}$, where $1/p + 1/p' = 1$, there exists a function $K_x \in L^{p'}_{1/\nu}$ such that

$$f(x) = \langle f, K_x \rangle = \int_{\mathbb{R}^d} f(t)\overline{K_x(t)}\, dt$$

for all $f \in V^p_\nu(\phi)$. In addition, it will be shown that $K_x \in V^{p'}_{1/\nu}(\phi)$.

In the case of a Hilbert space $\mathcal{H}$ of continuous functions on $\mathbb{R}^d$, such as $V^2(\phi)$, the following terminology is used. A Hilbert space is an RKHS [117] if, for any $x \in \mathbb{R}^d$, the pointwise evaluation $f \to f(x)$ is a bounded linear functional on $\mathcal{H}$. The unique functions $K_x \in \mathcal{H}$ satisfying $f(x) = \langle f, K_x \rangle$ are called the reproducing kernels of $\mathcal{H}$.

With this terminology we have the following consequence of Theorem 3.1.

THEOREM 4.1. *Let ν be ω-moderate. If $\phi \in W_0(L^1_\omega)$, then the evaluations $f \to f(x)$ are continuous functionals, and there exist functions $K_x \in V^1_\omega(\phi)$ such that $f(x) = \langle f, K_x \rangle$. The kernel functions are given explicitly by*

$$K_x(y) = \sum_{k\in\mathbb{Z}^d} \overline{\phi(x-k)}\ \widetilde{\phi}(y-k). \tag{4.1}$$

In particular, $V^2(\phi)$ is an RKHS.

The above theorem is a reformulation of Theorem 3.1. We only need to prove the formula for the reproducing kernel. Note that K_x in (4.1) is well defined: since $\tilde{\phi} \in W_0(L^1_\omega)$, Theorem 2.3 combined with Theorem 3.1(iii) implies that the sequence $\{\tilde{\phi}(x-k) \ : k \in \mathbb{Z}^d\}$ belongs to ℓ^1_ω. Thus, by the definition of $V^1_\omega(\phi)$, we have $K_x \in V^1_\omega(\phi)$, and so $K_x \in V^p_\nu(\phi)$ for any p with $1 \le p \le \infty$ and any ω-moderate weight ν. Furthermore, K_x is clearly the reproducing kernel, because if $f(x) = \sum_k c_k \phi(x-k)$, then

$$\langle f, K_x \rangle = \sum_{j,k} c_j \phi(x-k) \langle \phi(\cdot - j), \widetilde{\phi}(\cdot - k) \rangle = \sum_k c_k \phi(x-k) = f(x)\,.$$

4.2. Frames. In order to reconstruct a function $f \in V_\nu^p(\phi)$ from its samples $f(x_j)$, it is sufficient to solve the (infinite) system of equations

$$\sum_{k \in \mathbb{Z}^d} c_k \phi(x_j - k) = f(x_j) \tag{4.2}$$

for the coefficients (c_k). If we introduce the infinite matrix U with entries

$$U_{jk} = \phi(x_j - k) \tag{4.3}$$

indexed by $X \times \mathbb{Z}^d$, then the relation between the coefficient sequence c and the samples is given by

$$Uc = f|_X\,.$$

Theorem 3.1(ii) and (iii) imply that $f|_X \in \ell_\nu^p(X)$. Thus U maps $\ell_\nu^p(\mathbb{Z}^d)$ into $\ell_\nu^p(X)$.

Since $f(x) = \langle f, K_x \rangle$, the sampling inequality (3.2) implies that the set of reproducing kernels $\{K_{x_j}, x_j \in X\}$ spans $V_{1/\nu}^{p'}$. This observation leads to the following abstract concepts.

A *Hilbert frame* (or simply a *frame*) $\{e_j : \; j \in J\}$ of a Hilbert space $\mathcal{H}$ is a collection of vectors in $\mathcal{H}$ indexed by a countable set J such that

$$A\, \|f\|_{\mathcal{H}}^2 \leq \sum_j |\langle f, e_j \rangle|^2 \leq B\, \|f\|_{\mathcal{H}}^2 \tag{4.4}$$

for two constants $A, B > 0$ independent of $f \in \mathcal{H}$ [40].

More generally, a *Banach frame* for a Banach space B is a collection of functionals $\{e_j : \, j \in J\} \subset B^*$ with the following properties [54].

(a) There exists an associated sequence space B_d on the index set J, such that

$$A\|f\|_B \leq \|(\langle f, e_j \rangle)_{j \in J}\|_{B_d} \leq B\|f\|_B$$

for two constants $A, B > 0$ independent of $f \in B$.

(b) There exists a so-called reconstruction operator R from B_d into B, such that

$$R((\langle f, e_j \rangle)_{j \in J}) = f.$$

4.3. Relations between RKHSs, Frames, and Nonuniform Sampling. The following theorem translates the different terminologies that arise in the context of sampling theory [2, 40, 74].

THEOREM 4.2. *The following are equivalent:*

(i) $X = \{x_j : j \in J\}$ *is a set of sampling for* $V_\nu^p(\phi)$.

(ii) *For the matrix* U *in* (4.3), *there exist* $a, b > 0$ *such that*

$$a\,\|c\|_{\ell_\nu^p} \le \|Uc\|_{\ell_\nu^p(X)} \le b\,\|c\|_{\ell_\nu^p} \;\; \forall\, c \in \ell_\nu^p.$$

(iii) *There exist positive constants* $a > 0$ *and* $b > 0$ *such that*

$$a\,\|f\|_{L_\nu^p} \le \left(\sum_{x_j \in X} |f(x_j)|^p\,|\nu(x_j)|^p \right)^{1/p} \le b\,\|f\|_{L_\nu^p} \;\; \forall\, f \in V_\nu^p(\phi).$$

(iv) *For* $p = 2$, *the set of reproducing kernels* $\{K_{x_j} : x_j \in X\}$ *is a (Hilbert) frame for* $V^2(\phi)$.

REMARK 4.1.

(i) *The relation between RKHSs and uniform sampling of bandlimited functions was first reported by Yao* [117] *and used to derive interpolating series similar to* (1.1). *For the case of shift-invariant spaces, this connection was established in* [9]. *Sampling for functions in RKHSs was studied in* [84]. *For the general case of nonuniform sampling in shift-invariant spaces, the connection was established in* [5].

(ii) *The relation between Hilbert frames and sampling of bandlimited functions is well known* [14, 48]. *Sampling in shift-invariant spaces is more recent, and the relation between frames and sampling in shift-invariant spaces (with* $p = 2$ *and* $\nu = 1$*) can be found in* [5, 30, 75, 77, 102].

(iii) *The relation between Hilbert frames and the weighted average sampling mentioned in Remark* 3.1 *can be found in* [1]. *This relation is obtained via kernels that generalize the RKHS.*

5. Frame Algorithms for L_ν^p-Spaces. Theorem 4.2 states that a separated set $X = \{x_j : j \in J\}$ is a set of sampling for $V^2(\phi)$ if and only if the set of reproducing kernels $\{K_{x_j} : x_j \in X\}$ is a frame for $V^2(\phi)$. It is well known from frame theory that there exists a *dual frame* $\{\widetilde{K}_{x_j} : x_j \in X\} \subset V^2(\phi)$ that allows us to reconstruct the function $f \subset V^2(\phi)$ explicitly as

$$f(x) = \sum_{j \in J} \langle f, K_{x_j} \rangle \widetilde{K}_{x_j}(x) = \sum_{j \in J} f(x_j) \widetilde{K}_{x_j}(x). \tag{5.1}$$

However, a dual frame $\{\widetilde{K}_{x_j} : x_j \in X\}$ is difficult to find in general, and this method for recovering a function $f \in V^2(\phi)$ from its samples $\{f(x_j) :\; x_j \in X\}$ is often not practical.

Instead, the frame operator

$$\mathrm{T}\, f(x) = \sum_{j \in J} \langle f, K_{x_j} \rangle K_{x_j}(x) = \sum_{j \in J} f(x_j) K_{x_j}(x) \tag{5.2}$$

can be inverted via an iterative that we now describe. The operator $\mathrm{I} - \frac{2}{A+B}\,\mathrm{T}$ is contractive, i.e., the operator norm on $L^2(\mathbb{R}^d)$ satisfies the estimate

$$\left\| \mathrm{I} - \frac{2}{A+B}\,\mathrm{T} \right\|_{\mathrm{op}} \le \frac{B-A}{A+B} < 1\,,$$

where A, B are frame bounds for $\{K_{x_j} : x_j \in X\}$. Thus, $\frac{2}{A+B}\,\mathrm{T}$ can be inverted by the Neumann series

$$\frac{A+B}{2}\,\mathrm{T}^{-1} = \sum_{n=0}^{\infty}\left(\mathrm{I} - \frac{2}{A+B}\,\mathrm{T}\right)^{n}.$$

This analysis gives the *iterative frame reconstruction algorithm*, which is made up of an initialization step

$$f_1 = \sum_{j\in J} f(x_j) K_{x_j}$$

and iteration

$$f_n = \frac{2}{A+B} f_1 + \left(\mathrm{I} - \frac{2}{A+B}\,\mathrm{T}\right) f_{n-1}. \tag{5.3}$$

As $n \to \infty$, the iterative frame algorithm (5.3) converges to $f_\infty = \mathrm{T}^{-1} f_1 = \mathrm{T}^{-1}\,\mathrm{T} f = f$.

REMARK 5.1.

(i) *The computation of* T *requires the computation of the reproducing frame functions* $\{K_{x_j} :\ x_j \in X\}$*, which is a difficult task. Moreover, for each sampling set* X *we need to compute a new set of reproducing frame functions* $\{K_{x_j} :\ x_j \in X\}$.

(ii) *Even if the frame functions* $\{K_{x_j} :\ x_j \in X\}$ *are known, the performance of the frame algorithm depends sensitively on estimates for the frame bounds. Since accurate and explicit frame bounds, let alone optimal ones, are hardly ever known for nonuniform sampling problems, the frame algorithm converges very slowly in general. For efficient numerical computations involving frames, the primitive iteration* (5.3) *should therefore be replaced almost always by acceleration methods, such as Chebyshev or conjugate gradient acceleration. In particular, conjugate gradient methods converge at the optimal rate,* even without any knowledge of the frame bounds [49, 56].

(iii) *The convergence of the frame algorithm is guaranteed only in* L^2*, even if the function belongs to other spaces* L^p_ν*. It is a remarkable fact that in Hilbert space the norm equivalence* (4.4) *alone guarantees that the frame operator is invertible. In Banach spaces the situation is much more complicated and the existence of a reconstruction procedure must be postulated in the definition of a Banach frame. In the special case of sampling in shift-invariant spaces, the frame operator* T *is invertible on all* $V^p_\nu(\phi)$ *whenever* T *is invertible on* $V^2(\phi)$ *and* ϕ *possesses a suitable polynomial decay* [57].

6. Iterative Reconstruction Algorithms. Since the iterative frame algorithm is often slow to converge and its convergence is not even guaranteed beyond $V^2(\phi)$, alternative reconstruction procedures have been designed [4, 76]. These procedures are also iterative and based on a Neumann series. For the sake of exposition, the proofs of the results of this section and the next section are postponed to section 8.

The first step is to approximate the function f from its samples $\{f(x_j) :\ x_j \in X\}$ using an interpolation or a quasi-interpolation $\mathrm{Q_X} f$. For example, $\mathrm{Q_X} f$ could be a piecewise linear interpolation of the samples $f|_X$ or even an approximation by step functions, the so-called sample-and-hold interpolant.

The approximation $Q_X f$ is then projected in the space $V^p_\nu(\phi)$ to obtain the first approximation $f_1 = P\,Q_X f \in V^p_\nu(\phi)$. The error $e = f - f_1$ between the functions f and f_1 belongs to the space $V^p_\nu(\phi)$. Moreover, the values of e on the sampling set X can be calculated from $\{f(x_j) : x_j \in X\}$ and $(P\,Q_X f)(x_j)$. Then we repeat the interpolation-projection procedure on e and obtain a correction e_1. The updated estimate is now $f_2 = f_1 + e_1$. By repeating this procedure, we obtain a sequence $f_n = f_1 + e_1 + e_2 + e_3 + \cdots + e_{n-1}$ that converges to the function f.

In order to prove convergence results for this type of algorithm, we need the sampling set to be dense enough. The appropriate definition for the sampling density of X is again due to Beurling.

DEFINITION 6.1. *A set $X = \{x_j : j \in J\}$ is* γ_0-dense in $\mathbb{R}^d$ *if*

$$\mathbb{R}^d = \bigcup_j B_\gamma(x_j) \quad \forall\, \gamma > \gamma_0\,. \tag{6.1}$$

This definition implies that the distance of any sampling point to its nearest neighbor is at most $2\gamma_0$. Thus, strictly speaking, γ_0 is the inverse of a density; i.e., if γ_0 increases, the number of points per unit cube decreases. In fact, if a set X is γ_0-dense, then its Beurling density defined by (1.2) satisfies $D(X) \geq \gamma_0^{-1}$. This last relation states that γ_0-density imposes more constraints on a sampling set X than the Beurling density $D(X)$.

To create suitable quasi-interpolants, we proceed as follows. Let $\{\beta_j\}_{j\in J}$ be a partition of unity such that

(1) $0 \leq \beta_j \leq 1$ for all $j \in J$;
(2) supp $\beta_j \subset B_\gamma(x_j)$; and
(3) $\sum_{j\in J} \beta_j = 1$.

A partition of unity that satisfies these conditions is sometimes called a *bounded partition of unity.* Then the operator Q_X defined by

$$Q_X f = \sum_{j\in J} f(x_j)\beta_j$$

is a quasi-interpolant of the sampled values $f|_X$.

In this situation we have the following qualitative statement.

THEOREM 6.1. *Let ϕ in $W_0(L^1_\omega)$ and let* P *be a bounded projection from L^p_ν onto $V^p_\nu(\phi)$. Then there exists a density $\gamma > 0$ ($\gamma = \gamma(\nu, p, P)$) such that any $f \in V^p_\nu(\phi)$ can be recovered from its samples $\{f(x_j) : x_j \in X\}$ on any γ-dense set $X = \{x_j : j \in J\}$ by the iterative algorithm*

$$\begin{cases} f_1 = P\,Q_X f, \\ f_{n+1} = P\,Q_X(f - f_n) + f_n. \end{cases} \tag{6.2}$$

Then iterates f_n converge to f uniformly and in the $W(L^p_\nu)$- and L^p_ν-norms. The convergence is geometric, that is,

$$\|f - f_n\|_{L^p_\nu} \leq C\,\|f - f_n\|_{W(L^p_\nu)} \leq C'\,\|f - f_1\|_{L^p_\nu}\,\alpha^n,$$

for some $\alpha = \alpha(\gamma) < 1$.

The algorithm based on this iteration is illustrated in Figure 6.1. Figure 6.2 shows the reconstruction of a function f by means of this algorithm, and Figure 6.3 shows the reconstruction of an MRI image with missing data.

REMARK 6.1. *For $\nu = 1$, Theorem* 6.1 *was proved in* [4]. *For a special case of the weighted average sampling mentioned in Remark* 3.1, *a modified iterative algorithm and a theorem similar to Theorem* 6.1 *can be found in* [1].

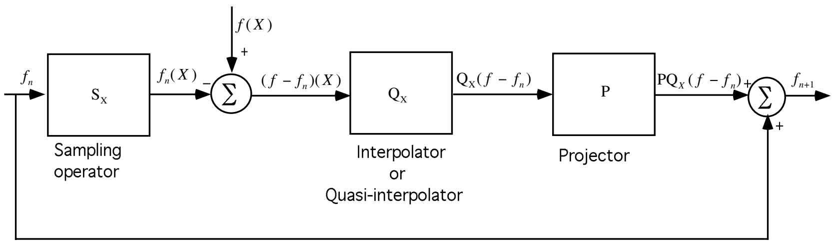

Fig. 6.1 *The iterative reconstruction algorithm of Theorem* 6.1.

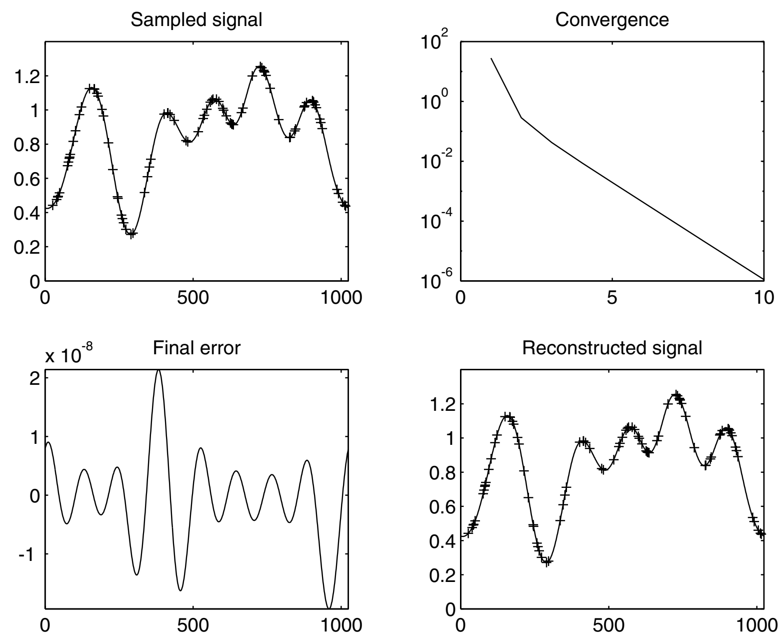

Fig. 6.2 *Reconstruction of a function f with $\|f\|_2 \approx 3.5$ using the iteration algorithm* (6.2) *of Theorem* 6.1. *Top left: Function f belonging to the shift-invariant space generated by the Gaussian function $e^{-x^2/2\sigma^2}$, $\sigma \approx 0.81$, and its sample values $\{f(x_j) : x_j \in X\}$ marked by + (density $\gamma \approx 0.8$). Top right: Error $\|f - f_n\|_{L^2}$ against the number of iterations. Bottom left: Final error $f - f_n$ after* 10 *iterations. Bottom right: Reconstructed function f_{10} (continuous line) and original samples $\{f(x_j) : x_j \in X\}$.*

Universal Projections in Weighted Shift-Invariant Spaces. Theorem 6.1 requires bounded projections from L^p_ν onto $V^p_\nu(\phi)$. In contrast to the situation in Hilbert space, the existence of bounded projections in Banach spaces is a difficult problem. In the context of nonuniform sampling in shift-invariant spaces, we would like the projections to satisfy additional requirements. In particular, we would like projectors that can be implemented with fast algorithms. Further, it would be useful to find a universal projection, i.e., a projection that works simultaneously for all L^p_ν, $1 \le p \le \infty$, and all weights ν. In shift-invariant spaces such universal projections do indeed exist.

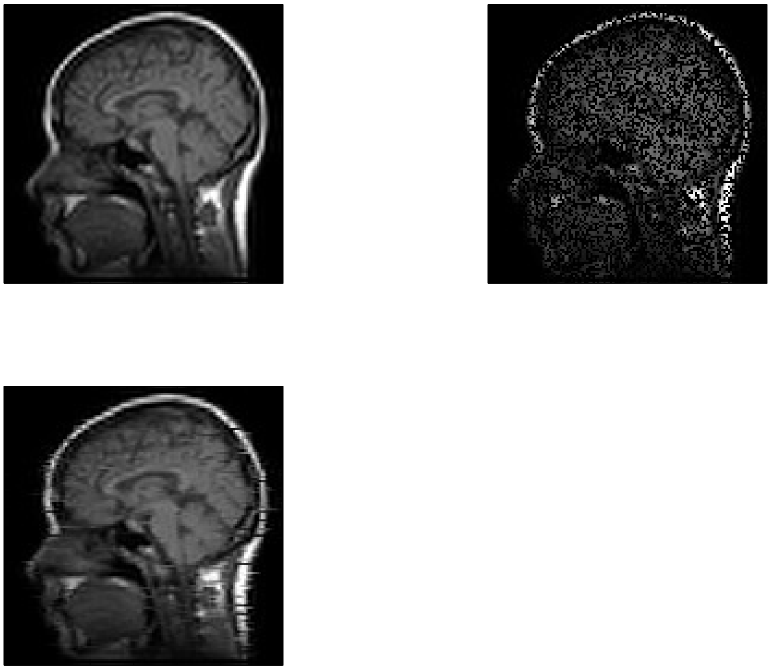

Fig. 6.3 *Missing data reconstruction. Top left: Original digital MRI image with* 128×128 *samples. Top right: MRI image with* 50% *randomly missing samples. Bottom left: Reconstruction using the iterative reconstruction algorithm* (6.2) *of Theorem* 6.1. *The corresponding shift-invariant space is generated by* $\phi(x, y) = \beta^3(x) \times \beta^3(y)$, *where* $\beta^3 = \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]} * \chi_{[0,1]}$ *is the B-spline function of degree* 3.

THEOREM 6.2. *Assume* $\phi \in W_0(L^1_\omega)$. *Then the operator*

$$\mathrm{P} : f \to \sum_{k \in \mathbb{Z}^d} \langle f, \tilde{\phi}(\cdot - k)\rangle \phi(\cdot - k)$$

is a bounded projection from L^p_ν *onto* $V^p_\nu(\phi)$ *for all* p, $1 \le p \le \infty$, *and all* ω*-moderate weights* ν.

REMARK 6.2. *The operator* P *can be implemented using convolutions and sampling. Thus the universal projector* P *can be implemented with fast "filtering" algorithms* [3].

7. Reconstruction in Presence of Noise. In practical applications the given data are rarely the exact samples of a function $f \in V^p_\nu(\phi)$. We assume more generally that f belongs to $W_0(L^p_\nu)$; then the sampling operator $f \mapsto \{f(x_j) : x_j \in X\}$ still makes sense and yields a sequence in $\ell^p_\nu(X)$. Alternatively, we may assume that $f \in V^p_\nu(\phi)$, but that the sampled sequence is a noisy version of $\{f(x_j) : x_j \in X\}$, e.g., that the sampling sequence has the form $\{f'_{x_j} = f(x_j) + \eta_j\} \in \ell^p_\nu(X)$. If a reconstruction algorithm is applied to noisy data, then the question arises whether the algorithm still converges, and if it does, to which limit it converges.

To see what is involved, we first consider sampling in the Hilbert space $V^2(\phi)$. Assume that $X = \{x_j : j \in J\}$ is a set of sampling for $V^2(\phi)$. Then the set of reproducing kernels $\{K_{x_j} : x_j \in X\}$ forms a frame for $V^2(\phi)$, and so $f \in V^2(\phi)$ can be reconstructed from the samples $f(x_j) = \langle f, K_{x_j}\rangle$ with the help of the dual frame $\{\widetilde{K}_{x_j} : x_j \in X\} \subset V^2(\phi)$ in the form of the expansion

$$f = \sum_{j\in J} \langle f, K_{x_j}\rangle \widetilde{K}_{x_j} = \sum_{j\in J} f(x_j)\widetilde{K}_{x_j}. \tag{7.1}$$

If $f \notin V^2(\phi)$, but $f \in W_0(L^2)$, say, then $f(x_j) \neq \langle f, K_{x_j}\rangle$ in general. However, the coefficients $\langle f, K_{x_j}\rangle$ still make sense for $f \in L^2$ and the frame expansion (7.1) still converges. The following result describes the limit of this expansion when $f \notin V^2(\phi)$.

THEOREM 7.1. *Assume that $X \subset \mathbb{R}^d$ is a set of sampling for $V^2(\phi)$ and let* P *be the orthogonal projection from L^2 onto $V^2(\phi)$. Then*

$$\mathrm{P}\, f = \sum_{j\in J} \langle f, K_{x_j}\rangle \widetilde{K}_{x_j}$$

for all $f \in L^2$.

The previous theorem suggests a procedure for sampling: the function f is first "prefiltered" with the reproducing kernel K_x to obtain the function f_a defined by $f_a(x) = \langle f, K_x\rangle$ for all $x \in \mathbb{R}^d$. Sampling f_a on X then gives a sequence of inner products $f_a(x_j) = \langle f, K_{x_j}\rangle$. The reconstruction (7.1) of f_a is then the least square approximation of f by a function $f_a \in V^2(\phi)$. In the case of bandlimited functions, we have $\phi(x) = \sin \pi x/(\pi x)$ and $K_x(t) = \frac{\sin \pi(t-x)}{\pi(t-x)}$. Then the inner product $f_a(x) = \langle f, K_x\rangle = f * \phi(x)$ is just a convolution. The filtering operation corresponds to a restriction of the bandwidth to $[-1/2, 1/2]$, because $(f * \phi)\hat{} = \hat{f} \cdot \chi_{[-1/2,1/2]}$, and is usually called prefiltering to reduce aliasing.

In practical situations, any sampling sequence is perturbed by noise. This perturbation can be modeled in several equivalent ways. (a) The function $f \in V^2(\phi)$ is sampled on X, and then noise $\eta_j \in \ell^2$ is added, resulting in a sequence $f'_j = f(x_j)+\eta_j$. (b) We start with an arbitrary sequence $f'_j \in \ell^2(X)$. (c) We sample a function $f \in W_0(L^2)$, which is not necessarily in $V^2(\phi)$.

In this situation, we wish to know what happens if we run the frame algorithm with the input sequence $\{f'_j : j \in J\}$. If $\{f'_j : j \in J\} \in \ell^2(X)$, we can still initialize the iterative frame algorithm by

$$g_1 = \sum_{j\in J} f'_j\, K_{x_j}. \tag{7.2}$$

This corresponds exactly to the first step in the iterative frame algorithm (5.3). Then we set

$$g_n = \frac{2}{A+B} g_1 + \left(I - \frac{2}{A+B} T\right) g_{n-1}\,. \tag{7.3}$$

Since $\{K_{x_j}\}$ is a frame for $V^2(\phi)$ by assumption, this iterative algorithm still converges in L^2, and its limit is

$$g_\infty = \lim_{n\to\infty} g_n = \sum_{j\in J} f'_j\ \widetilde{K}_{x_j}\,. \tag{7.4}$$

THEOREM 7.2. *Let X be a set of sampling for $V^2(\phi)$. Then for any $\{f'_j : j \in J\} \in \ell^2(X)$, the modified frame algorithm* (7.3) *with the initialization* (7.2) *converges*

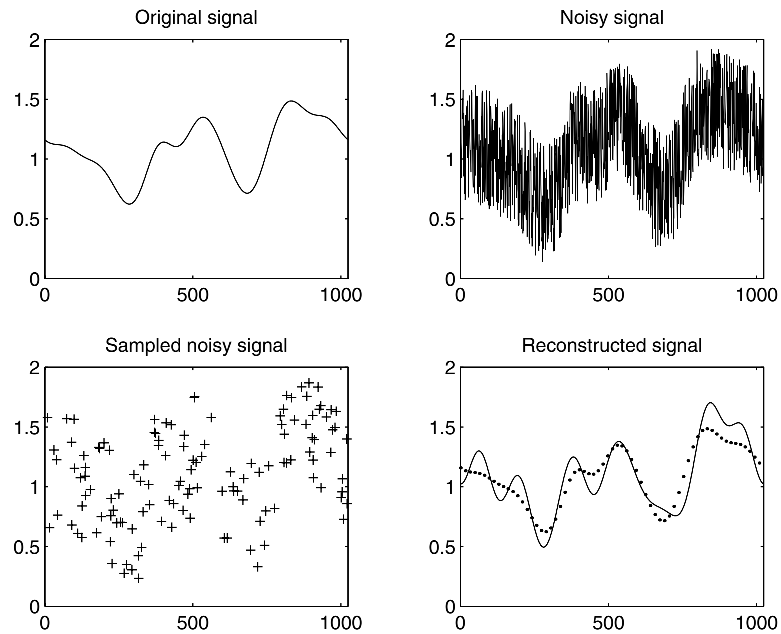

Fig. 7.1 *Reconstruction of a function f with additive noise using the iterative algorithm* (6.2) *of Theorem* 6.1*. Top left: Function f belonging to the shift-invariant space generated by the Gaussian function $e^{-x^2/2\sigma^2}$, $\sigma \approx 0.81$. Top right: Function f with an additive white noise ($SNR \approx 0db$). Bottom left: Noisy signal sampled on a nonuniform grid with maximal gap ≈ 0.51. Bottom right: Reconstructed function f_{10} after* 10 *iterations (continuous line) and original signal f (dotted line).*

to $g_\infty = \sum_{j\in\mathbb{Z}^d} f'_j \tilde{K}_{x_j} \in V^2(\phi)$. We have that

$$\sum_{j\in J} |f'_j - g_\infty(x_j)|^2 < \sum_{j\in J} |f'_j - g(x_j)|^2$$

for all $g \in V^2(\phi)$ with equality if and only if $g = g_\infty$. Thus g_∞ fits the given data optimally in the least squares sense.

Next we investigate the iterative algorithm (6.2) in the case of noisy samples $\{f'_j : j \in J\} \in \ell^p_\nu(X)$. We use the initialization

$$f_1 = \mathrm{P}\,\mathrm{Q_X}\, f' = \mathrm{P}\left(\sum_{j\in J} f'_j \beta_j\right), \tag{7.5}$$

and define the recursion as in (6.2) by

$$f_n = f_1 + (\mathrm{I} - \mathrm{P}\,\mathrm{Q_X}) f_{n-1}\,. \tag{7.6}$$

The convergence of this algorithm is clarified in the following theorem (see Figure 7.1).

Theorem 7.3. *Under the same assumptions as in Theorem* 6.1*, the algorithm* (7.6) *converges to a function $f_\infty \in V^p_\nu(\phi)$, which satisfies* $\mathrm{P}\,\mathrm{Q_X}\, f_\infty = \mathrm{P}\,\mathrm{Q_X}\{f'_j\}$.

8. Proofs of Lemmas and Theorems of Sections 6 and 7.

8.1. Proofs of Lemmas and Theorems of Section 6. To prove Theorems 6.1 and 6.2, we need the following lemmas.

LEMMA 8.1. *If $f \in V^p_\nu(\phi)$, then the oscillation (or modulus of continuity) $\operatorname{osc}_\delta(f)(x) = \sup_{|y|\le\delta} |f(x+y) - f(x)|$ belongs to $W(L^p_\nu)$. Moreover, for all $\epsilon > 0$, there exists $\delta_0 > 0$ such that*

$$\|\operatorname{osc}_\delta(f)\|_{W(L^p_\nu)} \le \epsilon \, \|f\|_{W(L^p_\nu)} \qquad \text{uniformly for all } f \in V^p_\nu(\phi) \text{ and } \delta < \delta_0 \,. \tag{8.1}$$

REMARK 8.1. *Inequality* (8.1) *implies that* $\operatorname{osc}_\delta$ *is a (sublinear) operator from* $V^p_\nu(\phi)$ *to* $W(L^p_\nu)$. *Using Theorems* 2.1(i) *and* 3.1(ii), *we conclude that* $\operatorname{osc}_\delta$ *is a (sublinear) operator from* $V^p_\nu(\phi)$ *to* L^p_ν, *and we also have* $\|\operatorname{osc}_\delta(f)\|_{L^p_\nu} \le C\epsilon \, \|f\|_{L^p_\nu}$ *for some constant* C *independent of* f *and* δ.

Proof. We show first that $\operatorname{osc}_\delta(\phi) \in W(L^1_\omega)$. Without loss of generality, assume $\delta \le 1$. Let $\mathcal{I} = [0,1]^d$, $\mathcal{C} = [-1,1]^d$, and $\mathcal{R} = \mathcal{I} + \mathcal{C} = [-1,2]^d$. Then for $j \in \mathbb{Z}^d$ we have

$$\begin{aligned}\sup_{x\in\mathcal{I}} \sup_{|y|\le\delta} |\phi(x+y+j)| &\le \sup_{x\in\mathcal{R}} |\phi(x+j)| \\ &\le \sum_{k\in\mathcal{R}\cap\mathbb{Z}^d} \sup_{x\in\mathcal{I}} |\phi(x+j+k)|.\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{x\in\mathcal{I}} |\operatorname{osc}_\delta(\phi)(x+j)| &\le \sup_{x\in\mathcal{I}} \sup_{|y|\le\delta} |\phi(x+y+j)| + \sup_{x\in\mathcal{I}} \sup_{|y|\le\delta} |\phi(x+j)| \\ &\le 2 \sum_{k\in\mathcal{R}\cap\mathbb{Z}^d} \sup_{x\in\mathcal{I}} |\phi(x+j+k)| \,.\end{aligned}$$

Summing over j, we obtain

$$\|\operatorname{osc}_\delta(\phi)\|_{W(L^1_\omega)} \le 2C \; \#(\mathcal{R}\cap\mathbb{Z}^d) \, \|\phi\|_{W(L^1_\omega)} \,. \tag{8.2}$$

Thus, $\operatorname{osc}_\delta(\phi) \in W(L^1_\omega)$.

Next we show that $\lim_{\delta\to 0} \operatorname{osc}_\delta(\phi)_{W(L^1_\omega)} = 0$. Since $\operatorname{osc}_\delta(\phi) \in W(L^1_\omega)$, there exists an integer $L_0 > 0$ such that

$$\sum_{|k|\ge L_0} \sup_{x\in\mathcal{I}} |\operatorname{osc}_\delta(\phi)(x+k)| \, \omega(k) < \frac{\epsilon}{2} \,. \tag{8.3}$$

Moreover, since ϕ is continuous, there exists a $\delta_0 > 0$ such that

$$\sup_{x\in\mathcal{I}} \sup_{|y|\le\delta} |\phi(x+y+k) - \phi(x+k)| \, \omega(k) \le \frac{\epsilon}{(2L_0)^d} \tag{8.4}$$

for all $|k| < L_0$ and all $\delta < \delta_0$.

Combining (8.3) and (8.4), we obtain that for any $\epsilon > 0$ there exists a $\delta_0 > 0$ such that

$$\|\operatorname{osc}_\delta(\phi)\|_{W(L^1_\omega)} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \quad \forall\, \delta,\ 0 < \delta \le \delta_0.$$

Thus, $\|\operatorname{osc}_\delta(\phi)\|_{W(L^1_\omega)} \to 0$ as $\delta \to 0$.

Finally, if $f = \sum_{k\in\mathbb{Z}^d} c_k\phi(\cdot - k) \in V^p_\nu(\phi)$, then we have

$$\begin{aligned}
\mathrm{osc}_\delta(f)(x) &= \sup_{|y|\le\delta} \left| \sum_{k\in\mathbb{Z}^d} c_k \left(\phi(x-k) - \phi(x+y-k)\right) \right| \\
&\le \sum_{k\in\mathbb{Z}^d} |c_k| \sup_{|y|\le\delta} |\phi(x-k) - \phi(x+y-k)| \\
&\le \sum_{k\in\mathbb{Z}^d} |c_k| \,\mathrm{osc}_\delta(\phi)(x-k).
\end{aligned}$$

Therefore Lemma 2.9 implies that

$$\|\mathrm{osc}_\delta(f)\|_{W(L^p_\nu)} \le C\,\|c\|_{\ell^p_\nu}\,\|\mathrm{osc}_\delta(\phi)\|_{W(L^1_\omega)}\,,$$

so (8.1) follows. □

Given a bounded uniform partition of unity $\{\beta_j\}$ associated with a separated sampling set X, we define a quasi-interpolant $\mathrm{Q_X}\,c$ on sequences by

$$\mathrm{Q_X}\,c = \sum_{j\in J} c_j\beta_j\,.$$

If $f \in W_0(L^p_\nu)$, we write

$$\mathrm{Q_X}\,f = \sum_{j\in J} f(x_j)\beta_j$$

for the quasi-interpolant of the sequence $c_j = f(x_j)$. If the partition of unity satisfies the additional condition $\beta_j(x_j) = 1$, hence $\beta_j(x_k) = 0$ for $k \ne j$, then $\mathrm{Q_X}\,c(x_j) = c_j$ for all $j \in J$ and $\mathrm{Q_X}\,c$ actually interpolates the sequence c.

LEMMA 8.2. *If $\{\beta_j\}$ is a bounded uniform partition of unity, then $\mathrm{Q_X}$ is a bounded operator from $\ell^p_\nu(X)$ to L^p_ν and to $W(L^p_\nu)$, i.e., $\|\mathrm{Q_X}\,c\|_{W(L^p_\nu)} \le C\|c\|_{\ell^p_\nu(X)}$. In particular, if $f \in W_0(L^p_\nu)$, then*

$$\|\mathrm{Q_X}\,f\|_{L^p_\nu} \le \|\mathrm{Q_X}\,f\|_{W(L^p_\nu)} \le C\|f|_X\|_{\ell^p_\nu(X)} \le C'\|f\|_{W(L^p_\nu)}\,.$$

Proof. Let χ be the characteristic function of the compact set $B_\gamma(0) + [0,1]^d$. Since $0 \le \beta_j \le 1$ and supp $\beta_j \subset B_\gamma(x_j)$, we conclude that for all $x_j \in k + [0,1]^d$,

$$\beta_j(x) \le \chi(x-k)\,.$$

Therefore,

$$\left|\sum_{j\in J} c_j\beta_j\right| \le \sum_{k\in\mathbb{Z}^d}\Bigg(\sum_{j:x_j\in k+[0,1]^d} |c_j|\Bigg)\chi(\cdot - k),$$

and consequently Lemma 2.9 implies that

$$\Bigg\|\sum_{j\in J} c_j\beta_j\Bigg\|_{W(L^p_\nu)} \le C\Bigg(\sum_{k\in\mathbb{Z}^d}\Bigg(\sum_{j:x_j\in k+[0,1]^d}|c_j|\Bigg)^p \nu(k)^p\Bigg)^{1/p}\|\chi\|_{W(L^1_\omega)}.$$

Since X is separated, there are at most N sampling points x_j in each cube $k + [0,1]^d$. So by Hölder's inequality we have $\left(\sum_{j:x_j\in k+[0,1]^d}|c_j|\right)^p \le N^{p/p'}\sum_{j:x_j\in k+[0,1]^d}|c_j|^p$.

Since furthermore $\nu(k) \le C\nu(x_j)$ for $x_j \in k + [0,1]^d$ by Lemma 2.6, we have proved that

$$\left\| \sum_{j\in J} c_j \beta_j \right\|_{W(L^p_\nu)} \le C' \left(\sum_{j\in J} |c_j|^p \nu(x_j)^p \right)^{1/p} = C' \|c\|_{\ell^p_\nu} .$$

Now the boundedness of $\mathrm{Q_X}$ on $W_0(L^p_\nu)$ follows from

$$\left\| \sum_{j\in J} f(x_j)\beta_j \right\|_{W(L^p_\nu)} \le C' \|f|_X\|_{\ell^p_\nu(X)} \le C'' \|f\|_{W(L^p_\nu)} . \qquad \square$$

LEMMA 8.3. *Let* P *be any bounded projection from* L^p_ν *onto* $V^p_\nu(\phi)$*. Then there exists a* $\gamma_0 = \gamma_0(\mathrm{P})$ *such that the operator* $\mathrm{I} - \mathrm{P\,Q_X}$ *is a contraction on* $V^p_\nu(\phi)$ *for every separated* γ*-dense set* X *with* $\gamma \le \gamma_0$*.*

Proof. For $f \in V^p_\nu(\phi)$ we have

$$\begin{aligned} \|f - \mathrm{P\,Q_X} f\|_{L^p_\nu} &= \|\mathrm{P} f - \mathrm{P\,Q_X} f\|_{L^p_\nu} \\ &\le \|\mathrm{P}\|_{\mathrm{op}} \|f - \mathrm{Q_X} f\|_{L^p_\nu} \\ &\le \|\mathrm{P}\|_{\mathrm{op}} \|\mathrm{osc}_\gamma(f)\|_{L^p_\nu} \\ &\le C_1 \epsilon \|\mathrm{P}\|_{\mathrm{op}} \|f\|_{L^p_\nu} . \end{aligned}$$

We can choose γ so small that $C_1\epsilon \|\mathrm{P}\|_{\mathrm{op}} < 1$ to get a contraction. $\square$

REMARK 8.2. *Diligent bookkeeping shows that the sufficient sampling density is determined by the inequality*

$$C_1 C_2 \left\| \mathrm{T}^*_{\tilde\phi} \right\|_{op} \|\mathrm{osc}_\gamma(\phi)\|_{W(L^1_\omega)} \|\mathrm{P}\|_{op} < 1,$$

where $\|\mathrm{P}\|_{op}$ *is the operator norm of the projector* P *on* $V^p_\nu(\phi)$*,* C_1 *is the constant in* (2.9) *or* (2.10)*,* C_2 *is the constant in Theorem* 2.2(iii)*, and* $\| \mathrm{T}^*_{\tilde\phi} \|_{op}$ *is the operator norm in* (2.17)*.*

Proof (of Theorem 6.1). Let $e_n = f - f_n$ be the error after n iterations. By (6.2), the sequence e_n satisfies the recursion

$$\begin{aligned} e_{n+1} &= f - f_{n+1} \\ &= f - f_n - \mathrm{P\,Q_X}(f - f_n) \\ &= (I - \mathrm{P\,Q_X}) e_n . \end{aligned}$$

Using Lemma 8.3, we may choose γ so small that $\| \mathrm{I} - \mathrm{P\,Q_X} \|_{\mathrm{op}} = \alpha < 1$. Therefore, we obtain

$$\|e_{n+1}\|_{W(L^p_\nu)} \le \alpha \|e_n\|_{W(L^p_\nu)} \tag{8.5}$$

and

$$\|e_n\|_{W(L^p_\nu)} \le \alpha^n \|e_0\|_{W(L^p_\nu)} .$$

Thus $\|e_n\|_{W(L^p_\nu)} \to 0$, and the proof is complete. $\square$

Proof (of Theorem 6.2). Using the operators T_ϕ and $\mathrm{T}^*_{\widetilde{\phi}}$ defined in (2.15) and (2.16), we have

$$\mathrm{P}\, f = \sum_{k\in\mathbb{Z}^d} \langle f, \widetilde{\phi}(\cdot - k)\rangle \phi(\cdot - k) = \mathrm{T}_\phi\, \mathrm{T}^*_{\widetilde{\phi}} f\,.$$

The proof of Theorem 2.3 shows that $\mathrm{T}^*_{\widetilde{\phi}}\, \mathrm{T}_\phi = \mathrm{I}_{\ell^p_\nu}$, and Lemmas 2.9 and 2.10 show that $\mathrm{T}^*_{\widetilde{\phi}}\, \mathrm{T}_\phi$ is bounded from L^p_ν to $V^p_\nu(\phi)$. Therefore,

$$\mathrm{P}^2 = (\mathrm{T}_\phi\, \mathrm{T}^*_{\widetilde{\phi}})(\mathrm{T}_\phi\, \mathrm{T}^*_{\widetilde{\phi}}) = \mathrm{T}_\phi\, \mathrm{T}^*_{\widetilde{\phi}} = \mathrm{P},$$

and so P is a projection. Let $f \in V^p_\nu(\phi)$; then $f = \mathrm{T}_\phi\, c$ for some $c \in \ell^p_\nu$, so the range of P is $V^p_\nu(\phi)$. □

8.2. Proofs of Theorems of Section 7.

Proof (of Theorem 7.1). Let P be the orthogonal projection onto $V^2(\phi)$. Since $K_x \in V^2(\phi)$, we have $\mathrm{P}\, K_x = K_x$ for all $x \in \mathbb{R}^d$ and thus

$$\langle f, K_x\rangle = \langle f, \mathrm{P}\, K_x\rangle = \langle \mathrm{P}\, f, K_x\rangle$$

for all $f \in L^2$. Consequently,

$$\sum_{j\in J}\langle f, K_{x_j}\rangle \widetilde{K}_{x_j} = \sum_{j\in J}\langle \mathrm{P}\, f, K_{x_j}\rangle \widetilde{K}_{x_j} = \mathrm{P}\, f \tag{8.6}$$

because $\mathrm{P}\, f \in V^2(\phi)$ and (8.6) is the identity on $V^2(\phi)$. □

Proof (of Theorem 7.2). It is well known that the dual frame $\{\widetilde{K}_{x_j} :\ x_j \in X\}$ is given by $\widetilde{K}_{x_j} = \mathrm{T}^{-1}\, K_{x_j}$, where T is the frame operator defined by (5.2) [40]. Thus, the iteration (7.3) converges to

$$g_\infty = \mathrm{T}^{-1}\, g_1 = \sum_{j\in J} f'_j\ \mathrm{T}^{-1}\, K_{x_j} = \sum_{j\in J} f'_j\ \widetilde{K}_{x_j}. \tag{8.7}$$

To show the least squares property, we start with two simple observations. First, from (8.7) we see that

$$\sum_{j\in J} f'_j\, K_{x_j} = \mathrm{T} g_\infty\,,$$

and second, for $g \in V^2(\phi)$ we have

$$\begin{aligned} \sum_{j\in J} g(x_j)\overline{f'_j} &= \sum_{j\in nJ} \langle g, K_{x_j}\rangle \overline{f'_j} \\ &= \langle g, \sum_{j\in J} f'_j K_{x_j}\rangle \\ &= \langle g, \mathrm{T} g_\infty\rangle, \end{aligned} \tag{8.8}$$

and by definition of the frame operator, $\sum_j |g(x_j)|^2 = \langle g, \mathrm{T} g\rangle$. Using (8.8), we estimate the least square error as follows:

$$\begin{aligned} &\sum_{j\in J} |f'_j - g(x_j)|^2 - \sum_{j\in J} |f'_j - g_\infty(x_j)|^2 \\ &\quad = \sum_{j\in J}\left(|g(x_j)|^2 - 2\ \mathrm{Re}\ \ g(x_j)\overline{f'_j} - |g_\infty(x_j)|^2 + 2\ \mathrm{Re}\ \ g_\infty(x_j)\overline{f'_j}\right) \\ &\quad = \langle g, \mathrm{T} g\rangle - 2\mathrm{Re}\,\langle g, \mathrm{T} g_\infty\rangle - \langle g_\infty, \mathrm{T} g_\infty\rangle + 2\mathrm{Re}\,\langle g_\infty, \mathrm{T} g_\infty\rangle \\ &\quad = \langle (g - g_\infty), \mathrm{T}(g - g_\infty)\rangle > 0. \end{aligned}$$

The last expression is strictly positive for $g \neq g_\infty$, since T is both positive and invertible. ∎

Proof (of Theorem 7.3). The hypothesis of Theorem 6.1 guarantees that $\mathrm{I} - \mathrm{P}\,\mathrm{Q_X}$ is a contraction on $V^p_\nu(\phi)$. Therefore, the iterates f_n converge to some $f_\infty \in V^p_\nu(\phi)$. Taking limits in (7.6), we obtain

$$f_\infty = f_1 + (\mathrm{I} - \mathrm{P}\,\mathrm{Q_X}) f_\infty$$

or

$$f_1 = \mathrm{P}\,\mathrm{Q_X}\, f' = \mathrm{P}\,\mathrm{Q_X}\, f_\infty,$$

as desired. ∎

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