ANALISI MATEMATICA - LEZIONE 23

ESEMPI

- $$\int \frac{1}{x(x^2+2x+2)}\,dx = ?$$ $\Delta<0$

Sia
$$f(x)=\frac{1}{x(x^2+2x+2)}=\frac{A}{x}+\frac{Bx+C}{x^2+2x+2}$$

allora
$$A(x^2+2x+2)+x(Bx+C)=1 \iff \begin{cases} A+B=0 & x^2 \\ 2A+C=0 & x^1 \\ 2A=1 & x^0 \end{cases}$$

da cui $A=\frac{1}{2}$, $B=-\frac{1}{2}$, $C=-1$. Così

$$\int \frac{1}{x(x^2+2x+2)}\,dx = \int\left(\frac{1/2}{x}-\frac{1/2\,x+1}{x^2+2x+2}\right)dx$$

$$\overset{x=t-1}{=} \frac{1}{2}\log|x| - \int \frac{1/2(t-1)+1}{\underbrace{(t-1)^2+2(t-1)+2}_{t^2-\cancel{2t}+1+\cancel{2t}-\cancel{2}+\cancel{2}}}\,dt$$

$$=\frac{1}{2}\log|x|-\frac{1}{2}\int\frac{t+1}{t^2+1}\,dt$$

$$=\frac{1}{2}\log|x|-\frac{1}{2}\int\frac{t}{t^2+1}\,dt-\frac{1}{2}\int\frac{1}{t^2+1}\,dt$$

$$=\frac{1}{2}\log|x|-\frac{1}{4}\int\frac{1}{t^2+1}\,d(t^2+1)-\frac{1}{2}\int\frac{1}{t^2+1}\,dt$$

$$=\frac{1}{2}\log|x|-\frac{1}{4}\log(t^2+1)-\frac{1}{2}\operatorname{arctg}(t)+c$$

$$=\frac{1}{2}\log|x|-\frac{1}{4}\log(x^2+2x+2)-\frac{1}{2}\operatorname{arctg}(x+1)+c.$$

- $$\int \frac{1}{\operatorname{sen}(x)}\,dx = \int \frac{\operatorname{sen}(x)}{\operatorname{sen}^2(x)}\,dx \quad \leftarrow 1-\cos^2(x)$$

funz. razionale

$t = \cos(x)$

$dt = -\operatorname{sen}(x)\,dx$

$$= \int \frac{-dt}{1-t^2} = \int \left(\frac{\overset{=1/2}{A}}{t-1} + \frac{\overset{=-1/2}{B}}{t+1} \right) dt$$

$$= \frac{1}{2}\log|t-1| - \frac{1}{2}\log|t+1| + c$$

$$= \frac{1}{2}\log\left(\frac{1-\cos(x)}{1+\cos(x)}\right) + c.$$

- $$\int \frac{1}{x\log^2(x)(\log^2(x)+1)}\,dx$$

$t = \log(x)$

$dt = \frac{dx}{x}$

$$= \int \frac{1}{t^2(t^2+1)}\,dt = \int \left(\frac{\overset{=0}{A}}{t} + \frac{\overset{=1}{B}}{t^2} + \frac{\overset{=0}{C}t + \overset{=-1}{D}}{t^2+1} \right) dt$$

$$= -\frac{1}{t} - \operatorname{arctg}(t) + c = -\frac{1}{\log(x)} - \operatorname{arctg}(\log(x)) + c.$$

- $$\int \frac{e^{2x} + 2e^{-x}}{1+e^{-x}}\,dx$$

$t = e^x$

$x = \log(t)$

$dx = \frac{dt}{t}$

$$= \int \frac{t^2 + 2t^{-1}}{1+t^{-1}} \cdot \frac{dt}{t} = \int \frac{t^3+2}{t(t+1)}\,dt$$

$\frac{t^3+2}{t(t+1)} = t - 1 + \frac{t+2}{t(t+1)}$

$$= \int \left(t - 1 + \frac{\overset{=2}{A}}{t} + \frac{\overset{=-1}{B}}{t+1} \right) dt$$

$$= \frac{t^2}{2} - t + 2\log|t| - \log|t+1| + c$$

$$= \frac{e^{2x}}{2} - e^x + 2x - \log(e^x+1) + c.$$

- $$\int \frac{1+\sqrt{x}}{1+\sqrt{x}+x}\,dx$$

$t=\sqrt{x}$
$t^2=x$
$2t\,dt=dx$

$$\overset{\curvearrowright}{=} \int \frac{1+t}{1+t+t^2}\cdot 2t\,dt$$

$$=2\int\left(1-\frac{1}{1+t+t^2}\right)dt$$

$t=s-\frac{1}{2}$
$dt=ds$

$$\overset{\curvearrowright}{=} 2t-2\int\frac{1}{s^2+\left(\frac{\sqrt{3}}{2}\right)^2}\,ds$$

$$=2t-2\cdot\frac{2}{\sqrt{3}}\operatorname{arctg}\left(\frac{s}{\sqrt{3}/2}\right)+c$$

$$=2\sqrt{x}-\frac{4}{\sqrt{3}}\operatorname{arctg}\left(\frac{2\sqrt{x}+1}{\sqrt{3}}\right)+c.$$

- $$\int\frac{1}{\sqrt{1+e^x}}\,dx$$

$t=e^x;\ dx=\frac{dt}{t}$

$$\overset{\curvearrowright}{=} \int\frac{1}{\sqrt{1+t}}\cdot\frac{dt}{t}$$

$s=\sqrt{1+t}$
$s^2=1+t$
$2s\,ds=dt$

$$\overset{\curvearrowright}{=} \int\frac{1}{\cancel{s}}\cdot\frac{2\cancel{s}\,ds}{s^2-1}=\int\left(\frac{1}{s-1}-\frac{1}{s+1}\right)ds$$

$$=\log|s-1|-\log|s+1|+c$$

$$=\log\left(\frac{|s-1|}{|s+1|}\right)+c$$

$$=\log\left(\frac{\sqrt{1+e^x}-1}{\sqrt{1+e^x}+1}\right)+c.$$

OSSERVAZIONE

Se g' è continua in $[a,b]$ e f è continua in $g([a,b])$ allora

$$\int_a^b f(g(x))g'(x)dx \overset{t=g(x)}{=} \int_{g(a)}^{g(b)} f(t)dt.$$

$\boxed{\text{ESEMPI}}$

- $$\int_0^{\pi/4} \operatorname{sen}^2(x)\,dx = \frac{1}{2}\int_0^{\pi/4}(1-\cos(2x))dx \overset{t=2x,\ dt=2dx}{=} \frac{1}{2}\int_0^{\pi/2}(1-\cos(t))\frac{dt}{2}$$
$$= \frac{1}{4}\Big[t-\operatorname{sen}(t)\Big]_0^{\pi/2} = \frac{\pi}{8}-\frac{1}{4}$$

perché $\cos(2x)=\cos^2(x)-\operatorname{sen}^2(x)=1-2\operatorname{sen}^2(x)$ implica $\operatorname{sen}^2(x)=\frac{1}{2}(1-\cos(2x))$.

- $$\int_0^8 \frac{1}{1+\sqrt[3]{x}}dx = \int_0^2 \frac{1}{1+t}\cdot 3t^2 dt$$

$t=\sqrt[3]{x}$, $t^3=x$, $3t^2dt=dx$, $\sqrt[3]{0}=0$, $\sqrt[3]{8}=2$

$$= 3\int_0^2\left(t-1+\frac{1}{1+t}\right)dt = 3\left[\frac{t^2}{2}-t+\log|1+t|\right]_0^2$$
$$= 3\log(3).$$

$$\bullet \int_{\frac{\pi}{2}}^{\pi} \underbrace{\operatorname{sen}(2x)}_{2\operatorname{sen}(x)\cos(x)} \log(1-\cos(x))\,dx = -2\int_0^{-1} t\log(1-t)\,dt$$

$t=\cos(x)$, $dt=-\operatorname{sen}(x)dx$, $\cos(\frac{\pi}{2})=0$, $\cos(\pi)=-1$

$$=\int_{-1}^{0}\log(1-t)\,d(t^2)=\left[t^2\log(1-t)\right]_{-1}^{0}-\int_{-1}^{0}-\frac{t^2}{1-t}\,dt$$

$$=-\cancel{\log(2)}-\int_{-1}^{0}\left(t+1+\frac{1}{t-1}\right)dt$$

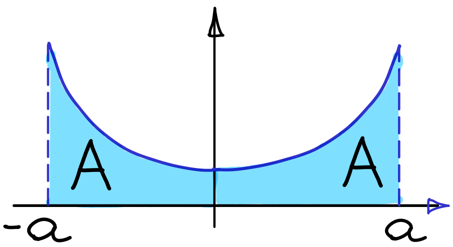
$$=-\cancel{\log(2)}-\left[\frac{t^2}{2}+t+\log|t-1|\right]_{-1}^{0}$$

$$=-\cancel{\log(2)}+\frac{1}{2}-1+\cancel{\log(2)}=-\frac{1}{2}.$$

OSSERVAZIONE

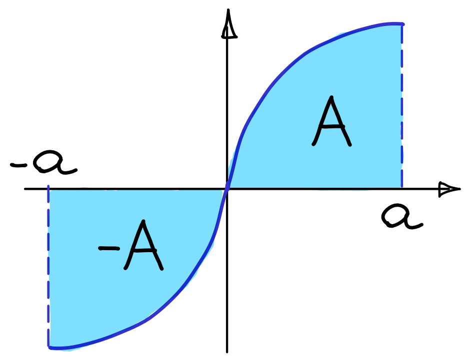
Se f è pari in $[-a,a]$ allora

$$\int_{-a}^{a}f(x)dx=2\int_{0}^{a}f(x)dx.$$



Se f è dispari in $[-a,a]$ allora

$$\int_{-a}^{a}f(x)dx=0.$$



- $\displaystyle\int_{-1}^{1} e^{-x^2}(|x|+\operatorname{sen}(x))\,dx = \underbrace{\int_{-1}^{1} e^{-x^2}|x|\,dx}_{\text{pari}} + \underbrace{\int_{-1}^{1} e^{-x^2}\operatorname{sen}(x)\,dx}_{\text{dispari}}$

$$= 2\int_{0}^{1} e^{-x^2}\underbrace{|x|}_{=x \text{ per } x\geq 0}\,dx + 0$$

$$= \left[-e^{-x^2}\right]_0^1 = 1-e^{-1}.$$

- $\displaystyle\int_{-3}^{3} \lfloor x \rfloor\,dx = -3-\cancel{2}-\cancel{1}+0+\cancel{1}+\cancel{2} = -3.$