ANALISI MATEMATICA 2 - LEZIONE 40

ESERCIZI DI RIEPILOGO - PARTE 3

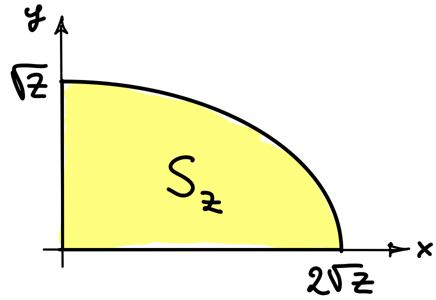
11 Calcolare $\displaystyle\lim_{r\to 0^+} \iiint_{D_r} xy\frac{\log(z)}{z^2}\,dx\,dy\,dz$

con $D_r = \{(x,y,z) : x\geq 0,\ y\geq 0,\ r\leq z\leq 2,\ x^2+4y^2\leq 4z\}$.

Integriamo per sezioni con $r\leq z\leq 2$

$$S_z = \{(x,y) : x^2+4y^2\leq 4z,\ x\geq 0,\ y\geq 0\}$$

$$\frac{x^2}{(2\sqrt{z})^2}+\frac{y^2}{(\sqrt{z})^2}\leq 1$$



Allora

$$\iiint_{D_r} xy\frac{\log(z)}{z^2}\,dx\,dy\,dz = \int_{z=r}^{2}\frac{\log(z)}{z^2}\left(\iint_{S_z} xy\,dx\,dy\right)dz$$

$\begin{cases} x = 2\rho\cos\theta \\ y=\rho\sin\theta\end{cases}$ $\vec{\Phi}(\rho,\theta)$

$$= \int_{z=r}^{2}\frac{\log(z)}{z^2}\int_{\rho=0}^{\sqrt{z}}\int_{\theta=0}^{\pi/2} 2\rho\cos\theta\cdot\rho\sin\theta\cdot\underbrace{(2\rho)}_{|\det J_{\Phi}|}\,d\rho\,d\theta$$

$$= 4\int_{z=r}^{2}\frac{\log(z)}{z^2}\left[\frac{\rho^4}{4}\right]_0^{\sqrt{z}}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}dz$$

$$= \frac{1}{2}\int_{z=r}^{2}\frac{\cancel{z^2}\log(z)}{\cancel{z^2}}\,dz = \frac{1}{2}\Big[z\log(z)-z\Big]_r^2$$

$$= \log(2)-1-\frac{1}{2}(r\log(r)-r) \xrightarrow{r\to 0^+} \log(2)-1.$$

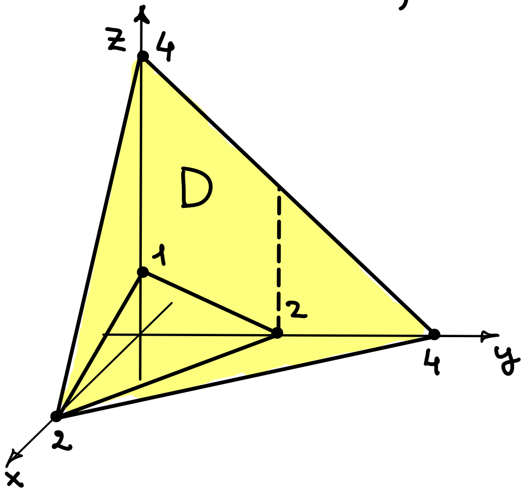
$\boxed{12}$ Calcolare $\iiint_D |x-1|\,dx\,dy\,dz$ con

$$D=\left\{(x,y,z):\ x+y+2z\geq 2,\ 2x+y+z\leq 4,\ x\geq 0,\ y\geq 0,\ z\geq 0\right\}.$$

D è il poliedro di vertici:

$(2,0,0),\ (0,4,0),\ (0,0,4)$ sul piano $2x+y+z=4$

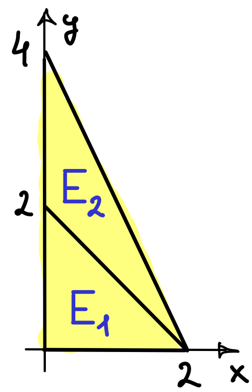
$(2,0,0),\ (0,2,0),\ (0,0,1)$ sul piano $x+y+2z=2$



Integrazione per fili:

$$\iiint_D |x-1|\,dx\,dy\,dz = \int_{x=0}^{2} |x-1|\left(\int_{y=0}^{2-x}\left(\int_{z=1-\frac{x+y}{2}}^{4-2x-y} dz\right)dy\right)dx \quad E_1$$

$$+\int_{x=0}^{2} |x-1|\left(\int_{y=2-x}^{4-2x}\left(\int_{z=0}^{4-2x-y} dz\right)dy\right)dx \quad E_2$$



$$= \dots = \int_0^2 |x-1|\left(5-5x+\frac{5}{4}x^2\right)dx + \int_0^2 |x-1|\left(2-2x+\frac{1}{2}x^2\right)dx$$

$$= \int_0^2 |x-1|\underbrace{\left(7-7x+\frac{7}{4}x^2\right)}_{\frac{7}{4}(4-4x+x^2)}dx = \frac{7}{4}\int_0^2 |x-1|(x-2)^2\,dx$$

$t=x-1$

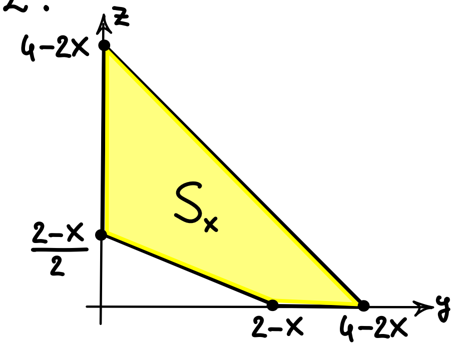
$$= \frac{7}{4}\int_{-1}^{1} \underbrace{|t|}_{p}\,\underbrace{(t-1)^2}_{t^2-2t+1}\,dt = \frac{\cancel{7}}{\cancel{4}_2}\cdot \cancel{2}\int_0^1 \left(t^3+t\right)dt$$

(t^2: p, $-2t$: d, 1: p)

$$= \frac{7}{2}\left[\frac{t^4}{4}+\frac{t^2}{2}\right]_0^1 = \frac{21}{8}.$$

Integrazione per sezioni con $0 \le x \le 2$:

$$S_x = \{(y,z) : y+2z \ge 2-x,\ y+z \le 4-2x,\ y \ge 0, z \ge 0\}$$



$$\iiint_D |x-1|\,dx\,dy\,dz = \int_{x=0}^{2} |x-1| \left(\iint_{S_x} dy\,dz\right) dx$$

$$= \int_{x=0}^{2} |x-1| \cdot |S_x|\,dx = \int_{x=0}^{2} |x-1| \left(\frac{(4-2x)^2}{2} - \frac{(2-x)^2}{4}\right) dx$$

$\frac{7}{4}(x-2)^2$

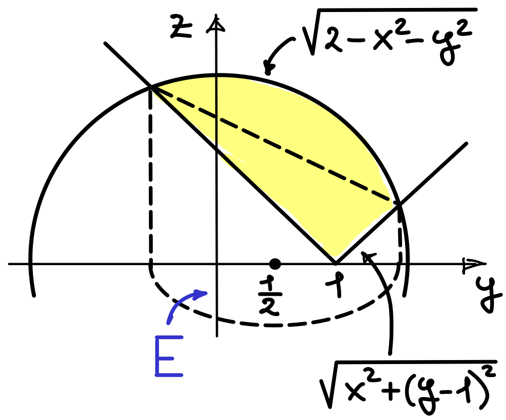
$$= \frac{7}{4}\int_0^2 |x-1|(x-2)^2\,dx = \cdots = \frac{21}{8}.$$

$\boxed{13}$ Calcolare $\iiint_D z\,dx\,dy\,dz$

con $D = \left\{(x,y,z) : x \ge 0,\ x^2+y^2+z^2 \le 2,\ z \ge \sqrt{x^2+(y-1)^2}\right\}$.

Integrazione per fili:

$$\sqrt{x^2+(y-1)^2} \le z \le \sqrt{2-x^2-y^2}$$



con $(x,y) \in E$ ossia $x \ge 0$ e

$$x^2+y^2+(x^2+(y-1)^2) \le 2$$
$$2x^2+2y^2-2y+1 \le 2$$
$$x^2+y^2-y \le \frac{1}{2}$$
$$x^2+\left(y-\frac{1}{2}\right)^2 \le \frac{1}{4}+\frac{1}{2} = \frac{3}{4}.$$

Quindi E è il semicerchio di centro $(0,\frac{1}{2})$ e raggio $\frac{\sqrt{3}}{2}$ contenuto nel semipiano $x \ge 0$.

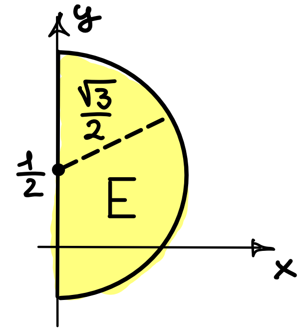
Allora

$$\iiint_D z\,dx\,dy\,dz = \iint_E \left(\int_{\sqrt{x^2+(y-1)^2}}^{\sqrt{2-x^2-y^2}} z\,dz\right) dx\,dy$$

$$= \frac{1}{2} \iint_E \left((2-x^2-y^2) - (x^2+(y-1)^2) \right) dx\,dy$$

$\hookrightarrow 1-2x^2-2y^2+2y = 1-2x^2-2(y-\frac{1}{2})^2+\frac{1}{2}$

$$= \iint_E \left(\frac{3}{4} - x^2 - (y-\frac{1}{2})^2 \right) dx\,dy$$



$\begin{cases} x = \rho\cos\theta \\ y = \frac{1}{2} + \rho\sin\theta \end{cases}$

$$= \int_{\rho=0}^{\frac{\sqrt{3}}{2}} \int_{\theta=-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{3}{4} - \rho^2 \right) \rho \, d\rho\, d\theta$$

$$= \pi \left[\frac{3}{4} \cdot \frac{\rho^2}{2} - \frac{\rho^4}{4} \right]_0^{\sqrt{3}/2} = \pi \left(\frac{3}{4} \right)^2 \cdot \left(\frac{1}{2} - \frac{1}{4} \right) = \frac{9\pi}{64}.$$

$\boxed{14}$ Calcolare la coordinata z del baricentro della curva

$$\vec{\gamma}(t) = \left(\frac{t^3}{3} + \frac{1}{t}, \sqrt{3}\,t, t \right) \text{ con } t \in [1,2].$$

Si ha che $\vec{\gamma}'(t) = \left(t^2 - \frac{1}{t^2}, \sqrt{3}, 1 \right)$ e quindi

$$\|\vec{\gamma}'(t)\| = \left(\left(t^2 - \frac{1}{t^2} \right)^2 + 3 + 1 \right)^{1/2} = \left(t^4 - 2 + \frac{1}{t^4} + 4 \right)^{1/2} = t^2 + \frac{1}{t^2}.$$

$\hookrightarrow t^4 + 2 + \frac{1}{t^4} = \left(t^2 + \frac{1}{t^2} \right)^2$

Lunghezza di γ:

$$|\gamma| = \int_\gamma ds = \int_1^2 \left(t^2 + \frac{1}{t^2} \right) dt = \left[\frac{t^3}{3} - \frac{1}{t} \right]_1^2 = \frac{8}{3} - \frac{1}{2} - \frac{1}{3} + 1 = \frac{17}{6}.$$

Coordinata z del baricentro:

$$\bar{z} = \frac{1}{|\gamma|} \int_\gamma z\, ds = \frac{6}{17} \int_1^2 t \left(t^2 + \frac{1}{t^2} \right) dt$$

$$= \frac{6}{17} \left[\frac{t^4}{4} + \log|t| \right]_1^2 = \frac{6}{17} \left(\frac{16-1}{4} + \log(2) \right)$$

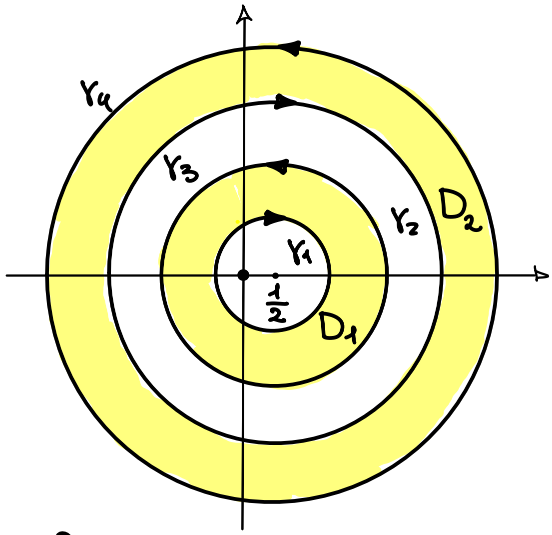
$$= \frac{45}{34} + \frac{6}{17} \log(2).$$

15 Calcolare $\sum_{i=1}^{4} \int_{\gamma_i} \langle \vec{F}, d\vec{s} \rangle$ dove

$$\vec{F}(x,y) = \left(-y\left(1+\frac{1}{x^2+y^2}\right), x\left(x+\frac{1}{x^2+y^2}\right)\right)$$

e $\gamma_1, \gamma_2, \gamma_3, \gamma_4$ sono le circonferenze di centro $(\frac{1}{2},0)$ e raggi rispettivamente $1,2,3,4$.
γ_1 e γ_3 sono percorse in senso orario mentre γ_2 e γ_4 sono percorse in senso antiorario.

Osserviamo che $\vec{F} = \vec{F}_1 + \vec{F}_2$ dove $\vec{F}_1 = (-y, x^2)$ e $\vec{F}_2 = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\right)$.
$\vec{F}_2$ è irrotazionale in $\mathbb{R}^2 \setminus \{(0,0)\}$



In alternativa al calcolo diretto applichiamo la formula di Gauss-Green a D_1 e D_2 indicati in figura.

$\vec{F}_2$ è irrotazionale

$$\frac{\partial}{\partial x}\left(x+\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-y-\frac{y}{x^2+y^2}\right) \overset{\downarrow}{=} \frac{\partial(x^2)}{\partial x} - \frac{\partial(-y)}{\partial y} = 2x+1.$$

Quindi, dato che il baricentro di D_1 è $(\frac{1}{2},0)$,

$$\sum_{i=1}^{2} \int_{\gamma_i} \langle \vec{F}, d\vec{s} \rangle = \iint_{D_1} (2x+1)\,dx\,dy = (2\bar{x}+1)|D_1|$$
$$= \left(2\cdot\frac{1}{2}+1\right)\cdot\left(\pi 2^2 - \pi 1^2\right) = 6\pi.$$

Analogamente per D_2

$$\sum_{i=3}^{4} \int_{\gamma_i} \langle \vec{F}, d\vec{s} \rangle = \iint_{D_2} (2x+1)\,dx\,dy = (2\bar{x}+1)|D_2|$$
$$= \left(2\cdot\frac{1}{2}+1\right)\cdot\left(\pi 4^2 - \pi 3^2\right) = 14\pi.$$

Così il risultato finale è

$$\sum_{i=1}^{4}\int_{\gamma_i}\langle\vec{F},d\vec{s}\rangle=\sum_{i=1}^{2}\int_{\gamma_i}\langle\vec{F},d\vec{s}\rangle+\sum_{i=3}^{4}\int_{\gamma_i}\langle\vec{F},d\vec{s}\rangle$$

$$=6\pi+14\pi=20\pi.$$

OSSERVAZIONE

Come visto a lezione, per $\vec{F}_2=\left(\frac{-y}{x^2+y^2},\frac{x}{x^2+y^2}\right)$, dato che ogni γ_i si avvolge intorno a $(0,0)$, si ha che

$$\int_{\gamma_1}\langle\vec{F}_2,d\vec{s}\rangle=\int_{\gamma_3}\langle\vec{F}_2,d\vec{s}\rangle=-2\pi$$

e

$$\int_{\gamma_2}\langle\vec{F}_2,d\vec{s}\rangle=\int_{\gamma_4}\langle\vec{F}_2,d\vec{s}\rangle=+2\pi.$$