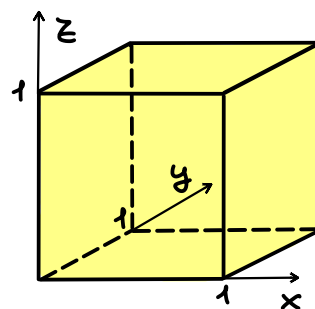


ANALISI MATEMATICA 2 - LEZIONE 37

ALCUNI ESERCIZI DEL FOGLIO 8

1.a Calcolare $\iint_S (x^2 + z^2) dS$ dove S è la superficie del cubo $[0, 1]^3$.



Per simmetria $\iint_S x^2 dS = \iint_S z^2 dS$.

$S = \bigcup_{i=1}^6 S_i$ è regolare a pezzi:

$$S_1 = \{(x, y, 0) : x, y \in [0, 1]\}, \quad S_2 = \{(x, y, 1) : x, y \in [0, 1]\}$$

e quindi

$$\iint_{S_1} x^2 dS = \iint_{S_2} x^2 dS = \int_0^1 \int_0^1 x^2 dx dy = \left[\frac{x^3}{3} \right]_0^1 \left[y \right]_0^1 = \frac{1}{3}.$$

Inoltre

$$S_3 = \{(x, 0, z) : x, z \in [0, 1]\}, \quad S_4 = \{(x, 1, z) : x, z \in [0, 1]\},$$

$$S_5 = \{(0, y, z) : y, z \in [0, 1]\}, \quad S_6 = \{(1, y, z) : y, z \in [0, 1]\}$$

e così

$$\iint_{S_3} x^2 dS = \iint_{S_4} x^2 dS = \int_0^1 \int_0^1 x^2 dx dz = \left[\frac{x^3}{3} \right]_0^1 \left[z \right]_0^1 = \frac{1}{3},$$

$$\iint_{S_5} x^2 dS = 0, \quad \iint_{S_6} x^2 dS = |S_6| = 1.$$

Infine

$$\iint_S (x^2 + z^2) dS = 2 \sum_{i=1}^6 \iint_{S_i} x^2 dS = 2 \left(4 \cdot \frac{1}{3} + 0 + 1 \right) = \frac{14}{3}.$$

1.b Calcolare $\iint_S x \sqrt{z^2 - 4x^2} dS$ dove S è
 $S = \{ (x, y, z) : z = 2\sqrt{x^2 + y^2}, (x-1)^2 + y^2 \leq 1 \}$.
 (come cilindro)

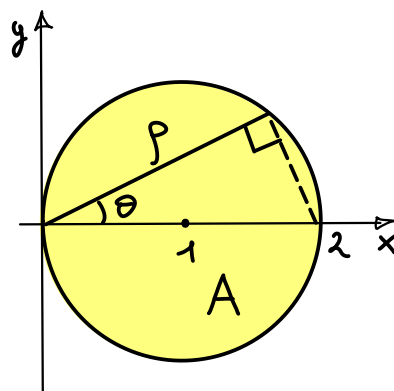
Parametrizzazione di S :

$$\vec{\sigma}(x, y) = (x, y, 2\sqrt{x^2 + y^2})$$

con $A = \{ (x, y) : (x-1)^2 + y^2 \leq 1 \}$.

Così

$$\vec{\sigma}_x \times \vec{\sigma}_y = \left(-\frac{2x}{\sqrt{x^2 + y^2}}, -\frac{2y}{\sqrt{x^2 + y^2}}, 1 \right).$$



Quindi $\|\vec{\sigma}_x \times \vec{\sigma}_y\| = \sqrt{5}$ e $\frac{2|y|}{\sqrt{x^2 + y^2}}$

$$\iint_S x \sqrt{z^2 - 4x^2} dS = \iint_A x \sqrt{4x^2 + 4y^2 - 4x^2} \cdot \sqrt{5} dx dy$$

$$\stackrel{CP}{=} 2\sqrt{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} p \cos\theta \cdot p |\sin\theta| p dp d\theta$$

$$\stackrel{CP}{=} 2\sqrt{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \cdot |\sin\theta| \left[\frac{p^4}{4} \right]_0^{2\cos\theta} d\theta$$

$$= 8\sqrt{5} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^5\theta \cdot |\sin\theta| d\theta \stackrel{\theta\text{-pari}}{=} 16\sqrt{5} \int_0^{\frac{\pi}{2}} \cos^5\theta d(-\cos\theta)$$

$$= 16\sqrt{5} \left[-\frac{\cos^6\theta}{6} \right]_0^{\frac{\pi}{2}} = \frac{8\sqrt{5}}{3}.$$

2.a Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove
 $\vec{F} = (3e^{x+y}, 2y-z, xy)$, $S = \partial D$ e

$$D = \{(x, y, z) : |x-1| + |y| \leq 1, 0 \leq z \leq x+y\}.$$

S è orientata verso l'esterno.

Applichiamo il teorema della divergenza:

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(3e^{x+y}) + \frac{\partial}{\partial y}(2y-z) + \frac{\partial}{\partial z}(xy) = 3e^{x+y} + 2$$

e allora

$$\iint_S \langle \vec{F}, d\vec{S} \rangle \stackrel{TD}{=} \iiint_D \operatorname{div}(\vec{F}) dx dy dz$$

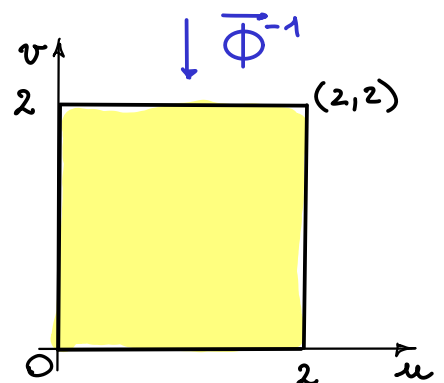
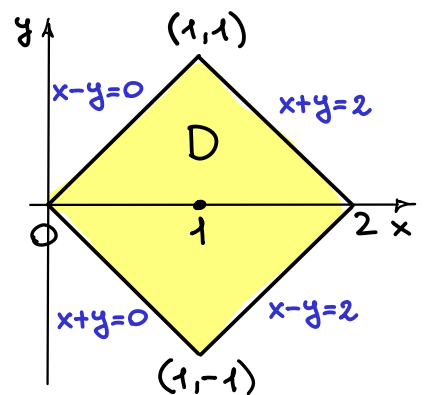
$$= \iint_{\{|x-1|+|y| \leq 1\}} (3e^{x+y} + 2) \left(\int_{z=0}^{x+y} dz \right) dx dy$$

$$= \iint_{\{|x-1|+|y| \leq 1\}} (3e^{x+y} + 2)(x+y) dx dy$$

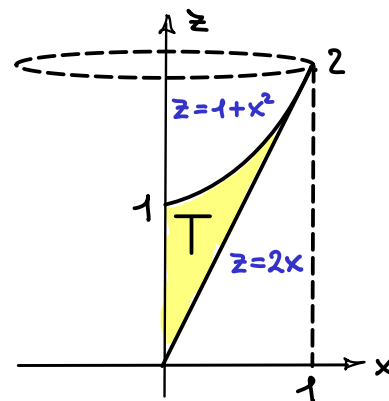
$$\begin{cases} u = x+y \\ v = x-y \end{cases} \quad \begin{matrix} \nearrow \\ \searrow \end{matrix} \quad \begin{matrix} | \det J_{\Phi} | \\ [0, 2]^2 = \Phi^{-1}(\{|x-1|+|y| \leq 1\}) \end{matrix}$$

$$= \int_0^2 (3e^u + 2)u du = \left[3e^u(u-1) + u^2 \right]_0^2$$

$$= 3e^2 + 4 - (-3) = 3e^2 + 7.$$



2.b Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove
 $\vec{F} = (e^{y^2}, 5x^2y, 5y^2z)$, $S = \partial D$ è orientata verso
 l'esterno e D è generato
 dalla rotazione completa di
 $T = \{(x, 0, z) : 0 \leq 2x \leq z \leq 1 + x^2 \leq 2\}$
 intorno all'asse z .



Applichiamo il teorema della divergenza:

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(e^{y^2}) + \frac{\partial}{\partial y}(5x^2y) + \frac{\partial}{\partial z}(5y^2z) = 5(x^2 + y^2).$$

e allora

$$\iint_S \langle \vec{F}, d\vec{S} \rangle \stackrel{TD}{=} \iiint_D \operatorname{div}(\vec{F}) \, dx \, dy \, dz$$

$$\stackrel{CC}{=} \int_{\rho=0}^1 5\rho^2 \left(\int_{z=2\rho}^{1+\rho^2} \left(\int_{\theta=0}^{2\pi} d\theta \right) dz \right) \rho \, d\rho = 10\pi \int_0^1 \rho^3 (1 + \rho^2 - 2\rho) \, d\rho$$

$$= 10\pi \left[\frac{\rho^4}{4} + \frac{\rho^6}{6} - 2\frac{\rho^5}{5} \right]_0^1 = 10\pi \cdot \frac{15 + 10 - 24}{60} = \frac{\pi}{6}.$$

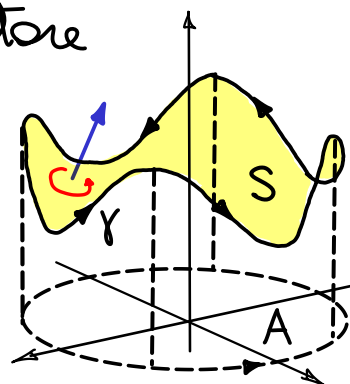
3.a

Usando il teorema del rotore calcolare $\int_Y \langle \vec{F}, d\vec{S} \rangle$ dove

$$\vec{F} = (x^2y, z(x-1), y(x+1)) \text{ e}$$

$$Y = \{(x, y, z) : x^2 + y^2 = 4, z = 2 + x^2y^2\}.$$

Y è orientata in modo che la sua proiezione su $z=0$ sia in verso antiorario.



$$\text{Sia } S = \{(x, y, z) : x^2 + y^2 \leq 4, z = 2 + x^2y^2\}.$$

Parametrizzazione cartesiana di S :

$$\vec{\sigma}(x, y) = (x, y, 2 + x^2y^2) \text{ con } A = \{(x, y) : x^2 + y^2 \leq 4\}.$$

Allora $\vec{\sigma}_x \times \vec{\sigma}_y = (-2xy^2, -2yx^2, 1)$ e $\vec{\sigma}$ induce su $Y = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\text{rot}(\vec{F}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y & z(x-1) & y(x+1) \end{bmatrix} = (2, -y, z).$$

Quindi

$$\int_Y \langle \vec{F}, d\vec{S} \rangle \stackrel{\text{TR}}{=} \iint_S \langle \text{rot}(\vec{F}), d\vec{S} \rangle$$

$$= \iint_A \langle (2, -y, z), (-2xy^2, -2yx^2, 1) \rangle dx dy$$

$$= \iint_{\{x^2+y^2 \leq 4\}} (-4xy^2 + 3x^2y^2 + 2) dx dy \stackrel{\text{CP}}{=} 3 \int_0^{2\pi} \int_0^2 \underbrace{\rho^4 \cos^2 \theta \cdot \frac{1}{4} \sin 2\theta}_{\text{x-dispari}} \rho d\rho d\theta + 2|A|$$

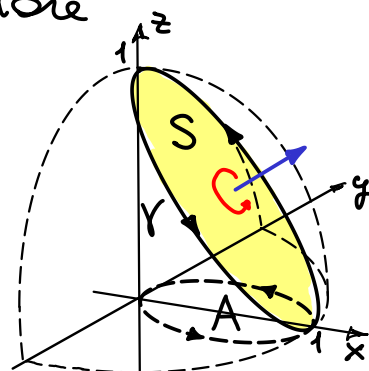
$$= \frac{3}{4} \int_0^{2\pi} \sin 2\theta d\theta \left[\frac{\rho^6}{6} \right]_0^2 + 8\pi = \frac{1}{4} \pi \cdot 32 + 8\pi = 16\pi.$$

3.6 Usando il teorema del rotore calcolare $\int_Y \langle \vec{F}, d\vec{S} \rangle$ dove

$$\vec{F} = (y + ye^{xy}, xe^{xy}, x^3) \text{ e}$$

$$Y = \{(x, y, z) : x^2 + y^2 + z^2 = 1, x + z = 1\}$$

orientata in modo che la sua proiezione su $z=0$ sia in verso antiorario.



$$\text{Sia } S = \{(x, y, z) : x^2 + y^2 + z^2 \leq 1, x + z = 1\}.$$

Parametrizzazione cartesiana di S :

$$\vec{\sigma}(x, y) = (x, y, 1 - x) \text{ con } A = \{(x, y) : x^2 + y^2 + (1 - x)^2 \leq 1\}.$$

Allora $\vec{\sigma}_x \times \vec{\sigma}_y = (1, 0, 1)$ e $\vec{\sigma}$ induce su $\gamma = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\text{rot}(\vec{F}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + ye^{xy} & xe^{xy} & x^3 \end{bmatrix} = (0, -3x^2, -1).$$

Quindi

$$\begin{aligned} \int_Y \langle \vec{F}, d\vec{S} \rangle &\stackrel{\text{TR}}{=} \iint_S \langle \text{rot}(\vec{F}), d\vec{S} \rangle \\ &= \iint_A \langle (0, -3x^2, -1), (1, 0, 1) \rangle dx dy \\ &= - \iint_A dx dy = -|A| = -\pi \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = -\frac{\pi}{2\sqrt{2}}. \end{aligned}$$

∂A è un'ellisse di semiasse $\frac{1}{2}$ e $\frac{1}{\sqrt{2}}$.

$$x^2 + y^2 + (1 - x)^2 = 1 \iff 2x^2 - 2x + y^2 = 0$$

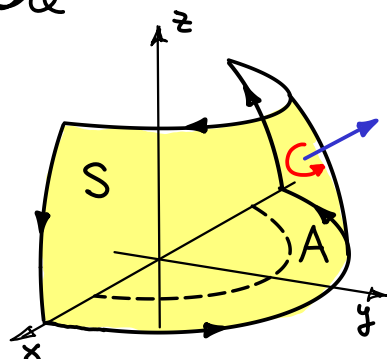
$$\iff 2(x - \frac{1}{2})^2 + y^2 = \frac{1}{2} \iff \frac{(x - \frac{1}{2})^2}{(\frac{1}{2})^2} + \frac{y^2}{(\frac{1}{\sqrt{2}})^2} = 1$$

4.a Usando il teorema del rotore calcolare $\int_{\partial^+ S} \langle \vec{F}, d\vec{S} \rangle$ dove

$$\vec{F} = (e^x, xy + 2x, x^2 + z^2) \text{ e}$$

$$S = \{(x, y, z) : x^2 + y^2 = 4 - z, 0 \leq z \leq 3, y \geq 0\}$$

orientata in modo che $\langle \vec{n}, \vec{F} \rangle \geq 0$.



Parametrizzazione cartesiana di S:

$$\vec{\sigma}(x, y) = (x, y, 4 - x^2 - y^2) \text{ con } A = \{(x, y) : 1 \leq x^2 + y^2 \leq 4, y \geq 0\}.$$

Allora $\vec{\sigma}_x \times \vec{\sigma}_y = (2x, 2y, 1)$ e $\vec{\sigma}$ induce su $\gamma = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\text{rot}(\vec{F}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & xy + 2x & x^2 + z^2 \end{bmatrix} = (0, -2x, y + 2).$$

Quindi

$$\int_{\gamma} \langle \vec{F}, d\vec{S} \rangle \stackrel{\text{TR}}{=} \iint_S \langle \text{rot}(\vec{F}), d\vec{S} \rangle$$

$$= \iint_A \langle (0, -2x, y + 2), (2x, 2y, 1) \rangle dx dy$$

$$= \iint_A (-4xy + y + 2) dx dy$$

simmetrico
per $x=0$

$$\stackrel{\text{CP}}{=} \int_{\theta=0}^{\pi} \int_{\rho=1}^2 \rho \sin \theta \rho d\rho d\theta + 2|A| \stackrel{=(4\pi-\pi)/2}{=}$$

$$= \left[-\cos \theta \right]_0^{\pi} \cdot \left[\frac{\rho^3}{3} \right]_1^2 + 3\pi = 2 \frac{8-1}{3} + 3\pi = \frac{14}{3} + 3\pi.$$