

# ANALISI MATEMATICA 2 - LEZIONE 27

## ALCUNI ESERCIZI DEL FOGLIO 6

**1.a** Calcolare il baricentro di

$$D = \left\{ (x, y) : \frac{h}{b^2} x^2 \leq y \leq h \right\}$$

con  $h, b > 0$ .

Per simmetria

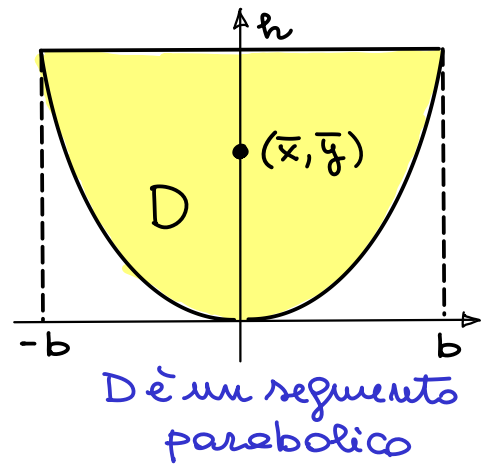
$$\bar{x} = \frac{1}{|D|} \iint_D x \, dx \, dy = 0$$

Calcolo di  $|D|$ :

$$\begin{aligned} |D| &= \iint_D 1 \, dx \, dy = 2 \int_{x=0}^b \left( \int_{y=\frac{h}{b^2} x^2}^h dy \right) dx = 2 \int_0^b \left( h - \frac{h}{b^2} x^2 \right) dx \\ &= 2h \left[ x - \frac{x^3}{3b^2} \right]_0^b = 2h \left( b - \frac{b}{3} \right) = \frac{4hb}{3}. \end{aligned}$$

Calcolo di  $\bar{y}$ :

$$\begin{aligned} \bar{y} &= \frac{1}{|D|} \iint_D y \, dx \, dy = \frac{2}{|D|} \int_{x=0}^b \left( \int_{y=\frac{h}{b^2} x^2}^h y \, dy \right) dx \\ &= \frac{1}{|D|} \int_0^b \left[ y^2 \right]_{\frac{h}{b^2} x^2}^h dx = \frac{3}{4hb} \int_0^b \left( h^2 - \frac{h^2 x^4}{b^4} \right) dx \\ &= \frac{3h^2}{4hb} \left[ x - \frac{x^5}{5b^4} \right]_0^b = \frac{3h}{4b} \left( b - \frac{b}{5} \right) \\ &= \frac{3h}{4} \cdot \frac{4}{5} = \frac{3h}{5}. \end{aligned}$$



**1.d**

$$\lim_{R \rightarrow +\infty} \iint_{D_R} \frac{1}{(1+x^2+y^2)^\alpha} dx dy \quad \text{per } \alpha > 0$$

$$\text{dove } D_R = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq R^2\}.$$

$$\iint_{D_R} \frac{1}{(1+x^2+y^2)^\alpha} dx dy \stackrel{CP}{=} \int_{\theta=0}^{2\pi} \int_{\rho=0}^R \frac{1}{(1+\rho^2)^\alpha} \rho d\rho d\theta$$

$$t = 1 + \rho^2, \quad dt = 2\rho d\rho$$

$$\downarrow = \frac{2\pi}{2} \int_1^{1+R^2} \frac{1}{t^\alpha} dt = \begin{cases} \pi \left[ \frac{t^{-\alpha+1}}{-\alpha+1} \right]_1^{1+R^2} & \text{se } \alpha \neq 1 \\ \pi [\log(t)]_1^{1+R^2} & \text{se } \alpha = 1 \end{cases}$$

Quindi:

$$\lim_{R \rightarrow +\infty} \iint_{D_R} \frac{1}{(1+x^2+y^2)^\alpha} dx dy = \begin{cases} \frac{\pi}{\alpha-1} & \text{se } \alpha > 1 \\ +\infty & \text{se } \alpha \leq 1 \end{cases}$$

**1.e**

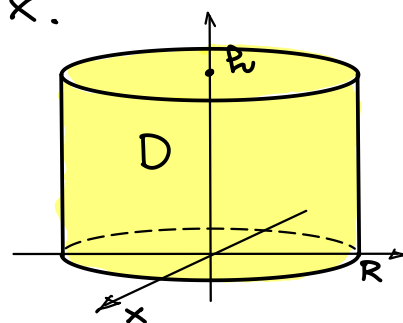
Calcolare  $I/M$  per  $D = \{(x, y, z) : x^2 + y^2 \leq R^2, 0 \leq z \leq h\}$   
con  $h, R > 0$  rispetto all'asse  $x$ .

La distanza da un punto  $(x, y, z)$  dell'asse  $x$  è  $\sqrt{y^2 + z^2}$ .

Quindi:

$$\frac{I}{M} = \frac{1}{|D|} \iiint_D (y^2 + z^2) dx dy dz$$

$$\stackrel{CC}{=} \frac{1}{\pi R^2 h} \int_{\rho=0}^R \int_{\theta=0}^{2\pi} \int_{z=0}^h (\rho^2 \sin^2 \theta + z^2) \rho d\rho d\theta dz$$



cilindro

$$\begin{aligned}
&= \frac{1}{\pi R^2 h} \int_{\rho=0}^R \int_{\theta=0}^{2\pi} \left[ \rho^3 \sin^2 \theta \cdot z + \rho \frac{z^3}{3} \right]_0^h d\rho d\theta \\
&= \frac{\cancel{h}}{\pi R^2 \cancel{h}} \int_{\rho=0}^R \rho^3 d\rho \cdot \int_{\theta=0}^{2\pi} \sin^2 \theta d\theta + \frac{\cancel{h}}{3\pi R^2 \cancel{h}} \int_{\rho=0}^R \rho d\rho \cdot \int_{\theta=0}^{2\pi} d\theta \\
&= \frac{1}{\pi R^2} \cdot \frac{R^4}{4} \cdot \pi + \frac{h^2}{3\pi R^2} \cdot \frac{R^2}{2} \cdot 2\pi = \frac{R^2}{4} + \frac{h^2}{3}.
\end{aligned}$$

1.j

Calcolare il volume di

$$D = \{(x, y, z) : x^2 + y^2 \leq 1, y^2 + z^2 \leq 1, x^2 + z^2 \leq 1\}.$$

$$\begin{aligned}
(x, y, z) \in D &\iff x^2 + y^2 \leq 1 \text{ e} \\
&|z| \leq \min(\sqrt{1-x^2}, \sqrt{1-y^2}) \\
&\quad \sqrt{1-x^2} \text{ se } |y| \leq |x|
\end{aligned}$$

Dato che  $D$  è simmetrico rispetto ai piani  $x=0$ ,  $y=0$ ,  $z=0$  e  $y=x$ .

$$|D| = 8 \cdot 2 |D_1| \text{ dove}$$

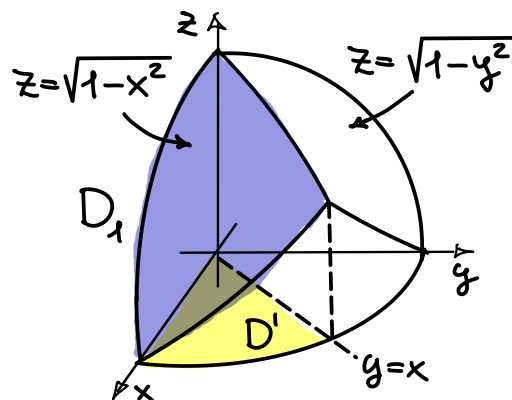
$$D_1 = \{(x, y, z) : (x, y) \in D', 0 \leq z \leq \sqrt{1-x^2}\} \text{ e}$$

$$D' = \{(x, y) : x \in [0, 1], 0 \leq y \leq x, 0 \leq y \leq \sqrt{1-x^2}\}.$$

Calcolo di  $|D_1|$ :

$$|D_1| = \iint_{D'} \sqrt{1-x^2} dx dy \stackrel{CP}{=} \int_{\theta=0}^{\pi/4} \int_{\rho=0}^1 \sqrt{1-\rho^2 \cos^2 \theta} \rho d\rho d\theta$$

$$\begin{aligned}
u &= 1 - \rho^2 \cos^2 \theta \\
du &= -2\rho \cos^2 \theta d\rho \quad \hookrightarrow \int \frac{1}{2\cos^2 \theta} \left( \int \sqrt{u} du \right) d\theta = \int_0^{\pi/4} \frac{1}{2\cos^2 \theta} \left[ \frac{2u^{3/2}}{3} \right]_{\sin^2 \theta}^1 d\theta
\end{aligned}$$



$$\begin{aligned}
&= \frac{1}{3} \int_0^{\pi/4} \frac{1 - \sin^3 \theta}{\cos^2 \theta} d\theta = \frac{1}{3} \int_0^{\pi/4} \frac{1}{\cos^2 \theta} d\theta + \frac{1}{3} \int_0^{\pi/4} \frac{1 - \cos^3 \theta}{\cos^2 \theta} d(\cos \theta) \\
&= \frac{1}{3} \left[ \tan \theta \right]_0^{\pi/4} + \frac{1}{3} \left[ -\frac{1}{\cos \theta} - \cos \theta \right]_0^{\pi/4} \\
&= \frac{1}{3} + \frac{1}{3} \left( -\sqrt{2} - \frac{\sqrt{2}}{2} + 2 \right) = 1 - \frac{\sqrt{2}}{2}.
\end{aligned}$$

Infine

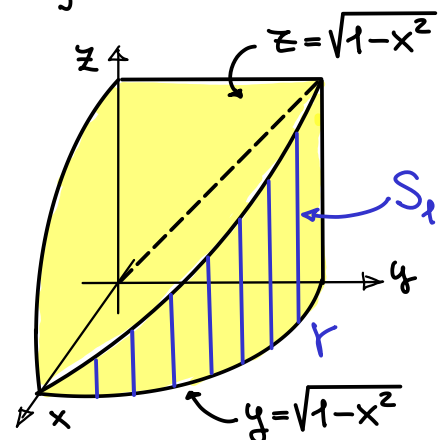
$$|D| = 16 |D_1| = 8(2 - \sqrt{2}).$$

**2.e**

Calcolare l'area della superficie di

$$D = \{(x, y, z) : x^2 + y^2 \leq 1, x^2 + z^2 \leq 1\}.$$

La superficie  $S$  del solido  $D$  è composta da 8 parti che, per le simmetrie rispetto ai piani  $x=0, y=0, z=0, y=z$ , hanno la stessa area. Una di queste parti è



$$S_1 = \{(x, y, z) : y = \sqrt{1-x^2}, 0 \leq z \leq \sqrt{1-x^2}, x \geq 0, y \geq 0\}.$$

$$\text{Quindi } |S_1| = \int_{\vec{r}} f d\sigma = \int_0^{\pi/2} f(\vec{r}(t)) \cdot \|\vec{r}'(t)\| dt$$

$$\text{con } \vec{r}(t) = (\cos t, \sin t) \text{ per } t \in [0, \frac{\pi}{2}] \text{ e } f(x, y) = \sqrt{1-x^2}.$$

$$\text{Così } \|\vec{r}'(t)\|^2 = \|(-\sin t, \cos t)\|^2 = (-\sin t)^2 + (\cos t)^2 = 1,$$

$$|S_1| = \int_0^{\pi/2} \underbrace{\sqrt{1-\cos^2 t}}_{|\sin t|} \cdot 1 dt = [-\cos t]_0^{\pi/2} = 1 \text{ e } |S| = 16 |S_1| = 16.$$

**3.2**

Calcolare  $\int_{\gamma} \langle \vec{F}, d\vec{s} \rangle$  dove  $\vec{F}(x,y) = (x+y, x-y)$   
e  $\vec{\gamma}(t) = (2\cos t, \sin t)$  per  $t \in [0, \frac{3\pi}{2}]$ .

$\vec{F} \in C^1(\mathbb{R}^2)$  ed è irrotazionale  $\frac{\partial F_1}{\partial y} = 1 = \frac{\partial F_2}{\partial x}$ .

Quindi  $\vec{F}$  è conservativo in  $\mathbb{R}^2$  (semplicemente connesso)

Potenziale: dobbiamo risolvere

Allora  $\frac{\partial U}{\partial x} = x+y$  e  $\frac{\partial U}{\partial y} = x-y$ .

$$U(x,y) = \int (x+y) dx = \frac{x^2}{2} + xy + C(y) \quad e$$

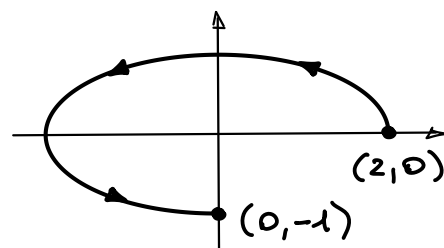
$$\frac{\partial U}{\partial y} = \frac{\partial}{\partial y} \left( \frac{x^2}{2} + xy + C(y) \right) = 0 + x + C'(y) \stackrel{?}{=} x-y \Rightarrow C'(y) = -y$$

$$\Rightarrow C(y) = -\frac{y^2}{2} + C. \quad \text{per il calcolo poniamo } C=0$$

Così  $U(x,y) = \frac{x^2}{2} + xy - \frac{y^2}{2} + C$  e

$$\int_{\gamma} \langle \vec{F}, d\vec{s} \rangle = U(\vec{\gamma}(\frac{3\pi}{2})) - U(\vec{\gamma}(0)) = -\frac{1}{2} - 2 = -\frac{5}{2}.$$

$(0, -1)$        $(2, 0)$

**3.6**

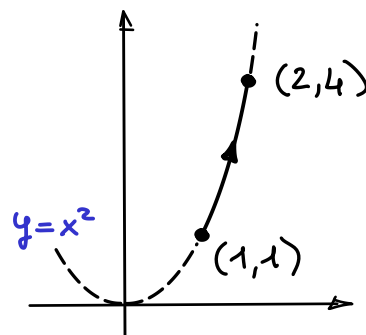
Calcolare  $\int_{\gamma} \langle \vec{F}, d\vec{s} \rangle$  dove  $\vec{F}(x,y) = (\frac{1}{x+y}, x)$

e  $\vec{\gamma}(t) = (t, t^2)$  per  $t \in [1, 2]$ .

Dato che  $\vec{\gamma}'(t) = (1, 2t)$ , ne ha

$$\begin{aligned} \int_{\gamma} \langle \vec{F}, d\vec{s} \rangle &= \int_1^2 \left( \frac{1}{t+t^2} + 2t \cdot t \right) dt \\ &= \left[ \log\left(\frac{t}{t+1}\right) + \frac{2t^3}{3} \right]_1^2 \end{aligned}$$

$$= \log\left(\frac{2}{3}\right) + \frac{16}{3} - \log\left(\frac{1}{2}\right) - \frac{2}{3} = \log\left(\frac{4}{3}\right) + \frac{14}{3}.$$

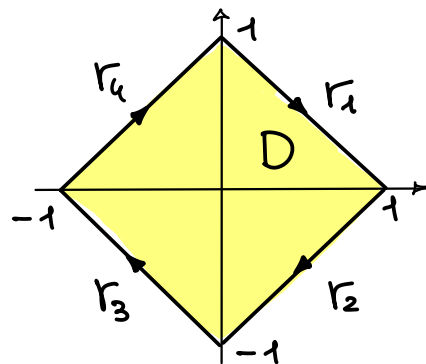


**3.d** Calcolare  $\int_{\gamma} \langle \vec{F}, d\vec{s} \rangle$  dove  $\vec{F}(x,y) = \left( \frac{x+y}{x+y+2}, -\frac{1}{x+y+2} \right)$

e  $\gamma$  è il bordo di

$$D = \{(x,y) : |x| + |y| \leq 1\}$$

percorso in senso orario.



Notiamo che in  $D$   $x+y+2 \neq 0$   
e quindi  $\vec{F}$  è continuo in  $D$

Parametrizzazione di  $\partial D$ : per  $t \in [0,1]$

$$\vec{\gamma}_1(t) = (t, 1-t) \quad \vec{\gamma}_2(t) = (1-t, -t)$$

$$\vec{\gamma}_3(t) = (-t, -1+t) \quad \vec{\gamma}_4(t) = (-1+t, t)$$

Ora si può fare il calcolo diretto con  $\vec{F}$  oppure  
osservare che  $\vec{F} = \vec{F}_1 + \vec{F}_2$ ,

$$\vec{F}_1(x,y) = \left( -\frac{1}{x+y+2}, 0 \right), \quad \vec{F}_2(x,y) = \left( 1 - \frac{1}{x+y+2}, -\frac{1}{x+y+2} \right)$$

dove  $\vec{F}_2$  è conservativo in  $D$  con potenziale

$$U(x,y) = x - \log(|x+y+2|). \text{ Allora}$$

$$\int_{\gamma} \langle \vec{F}, d\vec{s} \rangle = \int_{\gamma} \langle \vec{F}_1, d\vec{s} \rangle = \sum_{i=1}^4 \int_{\gamma_i} \langle \vec{F}_1, d\vec{s} \rangle$$

$$= \int_0^1 \frac{-1}{3} (+1) dt + \int_0^1 \frac{-1}{3-2t} (-1) dt + \int_0^1 \frac{-1}{1} (-1) dt + \int_0^1 \frac{-1}{2t+1} (+1) dt$$

$$= -\frac{1}{3} + \left[ -\frac{1}{2} \log|3-2t| \right]_0^1 + 1 - \left[ \frac{1}{2} \log|2t+1| \right]_0^1$$

$$= \frac{2}{3} + \cancel{\frac{1}{2} \log 3} - \cancel{\frac{1}{2} \log 3} = \frac{2}{3}.$$