

ANALISI MATEMATICA 2 - LEZIONE 21

ALCUNI ESERCIZI DEL FOGLIO 5

1.a $f(x, y, z) = x + z$ Max/min in
 $D = \left\{ (x, y, z) : \begin{array}{l} x^2 + y^2 + z^2 = 2 \\ z = x^2 + y^2 \end{array} \right\}$

$f \in C^2(\mathbb{R}^3)$, D è chiuso e limitato. Possiamo

$$g_1(x, y, z) = x^2 + y^2 + z^2 - 2 = 0, \quad g_2(x, y, z) = x^2 + y^2 - z = 0.$$

Regolarità:

$$\begin{bmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -1 \end{bmatrix}$$

Il rango non vale 2 se e solo se

$$\begin{cases} \det \begin{bmatrix} 2x & 2z \\ 2x & -1 \end{bmatrix} = 2x(-1-2z) = 0 \\ \det \begin{bmatrix} 2y & 2z \\ 2y & -1 \end{bmatrix} = 2y(-1-2z) = 0 \end{cases} \iff \begin{array}{l} (0, 0, z) \notin D \\ (x, y, -\frac{1}{2}) \notin D \end{array}$$

Tutti i punti di D sono regolari.

Moltiplicatori:

$$\begin{cases} 1 = \lambda 2x + \mu 2x \rightarrow 1 = 2x(\lambda + \mu) \rightarrow \lambda + \mu \neq 0 \\ 0 = \lambda 2y + \mu 2y \rightarrow 2y(\lambda + \mu) = 0 \rightarrow y = 0 \\ 1 = \lambda 2z - \mu \\ x^2 + y^2 + z^2 = 2 \\ z = x^2 + y^2 \end{cases}$$

Punti stazionari vincolati:

$$(1, 0, 1) \quad \lambda = \frac{1}{2}, \mu = 0$$

$$(-1, 0, 1) \quad \lambda = \frac{1}{6}, \mu = -\frac{2}{3}$$

$$\boxed{\begin{array}{l} f(1, 0, 1) = 2 \quad \text{Max} \\ f(-1, 0, 1) = 0 \quad \text{Min} \end{array}}$$

$$\begin{cases} y = 0 \\ 1 = 2(\lambda + \mu)x \\ 1 = 2\lambda z - \mu \\ x^2 + z^2 = 2 \rightarrow z + z^2 = 2 \\ z = x^2 \end{cases}$$

$z = 1 \rightarrow z = -2 = x^2 \quad \emptyset$
 $x = 1 \quad x = -1$

$(1, 0, 1)$ punto di max assoluto

$(-1, 0, 1)$ punto di min assoluto

2.a $\iint_D \frac{e^{x/y}}{y^4} dx dy \quad D = [0, 2] \times [1, 2]$

$$= \int_{y=1}^2 \frac{1}{y^4} \left(\int_{x=0}^2 e^{x/y} dx \right) dy = \int_{y=1}^2 \frac{1}{y^4} \left[y e^{x/y} \right]_0^2 dy = \int_1^2 \frac{e^{2/y} - 1}{y^3} dy$$

$t = \frac{1}{y} \rightarrow dt = -\frac{dy}{y^2}$

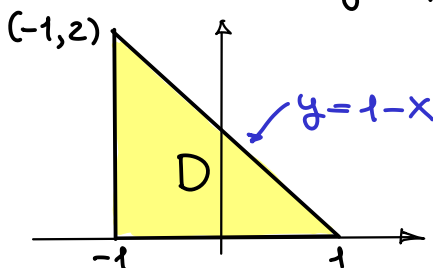
$$= \int_1^{1/2} \frac{e^{2t} - 1}{1/t} (-dt) = \int_{1/2}^1 (t e^{2t} - t) dt$$

$$= \left[\frac{t e^{2t}}{2} - \frac{e^{2t}}{4} - \frac{t^2}{2} \right]_{1/2}^1 = \frac{e^2}{2} - \frac{e^2}{4} - \frac{1}{2} - \left(\frac{e}{4} - \frac{e}{4} - \frac{1}{8} \right) = \frac{e^2}{4} - \frac{3}{8}$$

dove $\int t e^{2t} dt = \int t d\left(\frac{e^{2t}}{2}\right) = \frac{t e^{2t}}{2} - \frac{e^{2t}}{4} + c.$

2.b $\iint_D (1 - |x| - y) dx dy \quad D = \{(x, y) : x \geq -1, y \geq 0, x + y \leq 1\}$

$$= \int_{x=-1}^1 \left(\int_{y=0}^{1-x} (1 - |x| - y) dy \right) dx$$



$$= \int_{-1}^1 \left[(1 - |x|)y - \frac{y^2}{2} \right]_0^{1-x} dx = \int_{-1}^1 \left((1 - |x|)(1 - x) - \frac{1}{2}(1 - x)^2 \right) dx$$

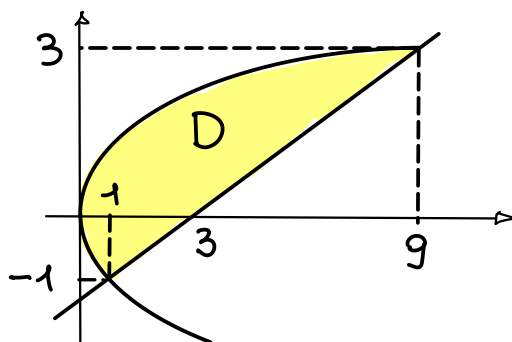
$\rightarrow \frac{1}{P} - \frac{|x|}{P} - \frac{x}{P} + \frac{x|x|}{d} - \frac{1}{2P} + \frac{x}{P} - \frac{x^2}{2P}$

$$= 2 \int_0^1 \left(\frac{1}{2} - x - \frac{x^2}{2} \right) dx = \left[x - x^2 - \frac{x^3}{3} \right]_0^1 = -\frac{1}{3}.$$

2.c

$$\iint_D \frac{1}{(2y-x+8)^2} dx dy \quad D = \{ (x,y) : y^2 \leq x \leq 2y+3 \}$$

$$= \int_{y=-1}^3 \left(\int_{x=y^2}^{2y+3} \frac{1}{(2y-x+8)^2} dx \right) dy$$



$$= \int_{y=-1}^3 \left[\frac{1}{2y-x+8} \right]_{y^2}^{2y+3} dy = \int_{-1}^3 \left(\frac{1}{5} - \frac{1}{2y-y^2+8} \right) dy$$

$-(y+2)(y-4)$

$$= \frac{3-(-1)}{5} + \frac{1}{6} \int_{-1}^3 \left(\frac{1}{y-4} - \frac{1}{y+2} \right) dy = \frac{4}{5} + \frac{1}{6} \left[\log \left| \frac{y-4}{y+2} \right| \right]_{-1}^3$$

$$= \frac{4}{5} + \frac{1}{6} \left(\log\left(\frac{1}{5}\right) - \log(5) \right) = \frac{4}{5} - \frac{\log(5)}{3}$$

3.a $D = \left\{ (x, y, z) : \frac{|x|}{4} + \frac{|y|}{3} + |z| \leq 1 \right\}$ Volume?

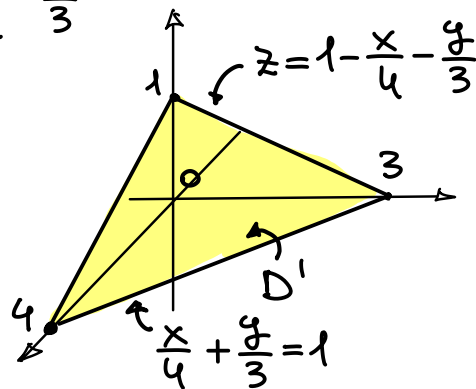
D è simmetrico rispetto ai piani $x=0, y=0, z=0$
 pertanto $|D| = 8|E|$ dove

$$E = \left\{ (x, y, z) : \frac{x}{4} + \frac{y}{3} + z \leq 1, x, y, z \geq 0 \right\}$$

1) Per fili con $\varphi_1(x, y) = 0, \varphi_2(x, y) = 1 - \frac{x}{4} - \frac{y}{3}$

$$\text{e } D' = \left\{ (x, y) : \frac{x}{4} + \frac{y}{3} \leq 1, x, y \geq 0 \right\}.$$

$$|D| = 8|E| = 8 \iint_{D'} \left(1 - \frac{x}{4} - \frac{y}{3} \right) dx dy$$



$$= 8 \int_{x=0}^4 \left(\int_{y=0}^{3-\frac{3x}{4}} \left(1 - \frac{x}{4} - \frac{y}{3} \right) dy \right) dx = 8 \int_{u=0}^1 \left(\int_{v=0}^{1-u} (1-u-v) 3 dv \right) 4 du$$

$\begin{cases} x=4u \\ y=3v \end{cases}$

$$= 8 \cdot 12 \int_0^1 \left[v - uv - \frac{v^2}{2} \right]_0^{1-u} du = 8 \cdot 12 \int_0^1 \left((1-u)^2 - \frac{(1-u)^2}{2} \right) du$$

$$= \frac{8 \cdot 12}{2} \int_0^1 (1-u)^2 du = 48 \left[-\frac{(1-u)^3}{3} \right]_0^1 = \frac{48}{3} = 16.$$

2) Per sezioni con $z \in [0, 1]$ e $S_z = \left\{ (x, y) : \frac{x}{4} + \frac{y}{3} \leq 1-z \right\}$

$$|D| = 8|E| = 8 \int_{z=0}^1 \left(\iint_{S_z} 1 dx dy \right) dz$$

triangolo rettangolo con
 cateti: $4(1-z)$ e $3(1-z)$

$$= 8 \int_0^1 |S_z| dz = 8 \int_0^1 \frac{4 \cdot 3 (1-z)^2}{2} dz = 8 \cdot 6 \left[-\frac{(1-z)^3}{3} \right]_0^1$$

$$= 8 \cdot \frac{1}{3} = 16.$$

3.6

$D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4, z \geq 1\}$ Volume?

calotta

1) Per fili con $D' = \{x^2 + y^2 + 1 \leq 4\}$

$\varphi_1(x, y) = 1$ e $\varphi_2(x, y) = \sqrt{4 - x^2 - y^2}$.

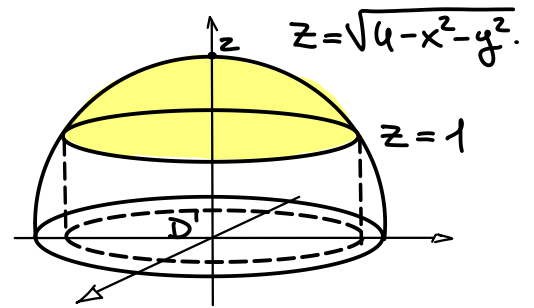
$$|D| = \iint_{D'} (\sqrt{4 - x^2 - y^2} - 1) dx dy$$

$$\stackrel{CP}{=} \int_{\rho=0}^{\sqrt{3}} \int_{\theta=0}^{2\pi} (\sqrt{4 - \rho^2} - 1) \rho d\rho d\theta = 2\pi \int_0^{\sqrt{3}} (\sqrt{4 - \rho^2} - 1) \rho d\rho$$

$t = 4 - \rho^2 \rightarrow dt = -2\rho d\rho$

$$= 2\pi \int_4^1 (t^{1/2} - 1) \left(-\frac{dt}{2}\right) = \pi \left[\frac{2}{3} t^{3/2} - t \right]_1^4$$

$$= \pi \left(\left(\frac{16}{3} - 4 \right) - \left(\frac{2}{3} - 1 \right) \right) = \frac{5}{3} \pi.$$



2) Per sezioni con $z \in [1, 2]$ e $S_z = \{(x, y) : x^2 + y^2 \leq 4 - z^2\}$
cerchio di raggio $\sqrt{4 - z^2}$

$$|D| = \int_{z=1}^2 \left(\iint_{S_z} 1 dx dy \right) dz = \int_1^2 |S_z| dz =$$

$$= \int_1^2 \pi(4 - z^2) dz = \pi \left[4z - \frac{z^3}{3} \right]_1^2 = \pi \left(8 - \frac{8}{3} - 4 + \frac{1}{3} \right)$$

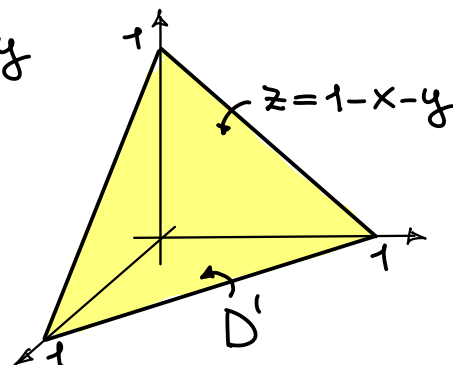
$$= \frac{5\pi}{3}$$

4.2

$$\iiint_D \frac{1}{(x+y+z+1)^3} dx dy dz$$

dove $D = \{(x, y, z) : x, y, z \geq 0, x+y+z \leq 1\}$.

Per fidi con $\varphi_1(x, y) = 0$, $\varphi_2(x, y) = 1 - x - y$
e $D' = \{(x, y) : x, y \geq 0, x+y \leq 1\}$



$$= \iint_{D'} \left(\int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3} dz \right) dx dy$$

$$= \iint_{D'} \left[\frac{-1/2}{(x+y+z+1)^2} \right]_0^{1-x-y} dx dy = \frac{1}{2} \iint_{D'} \left(\frac{1}{(x+y+1)^2} - \frac{1}{4} \right) dx dy$$

$$= \frac{1}{2} \int_{x=0}^1 \left(\int_{y=0}^{1-x} \frac{1}{(x+y+1)^2} dy \right) dx - \frac{|D'|}{8} = \frac{1}{2}$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{1}{x+y+1} \right]_0^{1-x} dx - \frac{1}{16} = \frac{1}{2} \int_0^1 \left(\frac{1}{x+1} - \frac{1}{2} \right) dx - \frac{1}{16}$$

$$= \frac{1}{2} \left[\log(x+1) \right]_0^1 - \frac{1}{4} - \frac{1}{16} = \frac{\log(2)}{2} - \frac{5}{16}.$$