

ANALISI MATEMATICA 2 - LEZIONE 20

INTEGRALI TRIPLI

Le definizioni e le proprietà viste per gli integrali doppi si possono estendere agli integrali tripli

$$\iiint_D f(x,y,z) dx dy dz$$

con $D \subseteq \mathbb{R}^3$ e $f: D \rightarrow \mathbb{R}$.

Si noti che $|D| = \iiint_D 1 dx dy dz$ indica il volume di D .

Vediamo i risultati principali utili per il calcolo.

TEOREMA (FORMULE DI RIDUZIONE)

Sia f continua in $D \subseteq \mathbb{R}^3$.

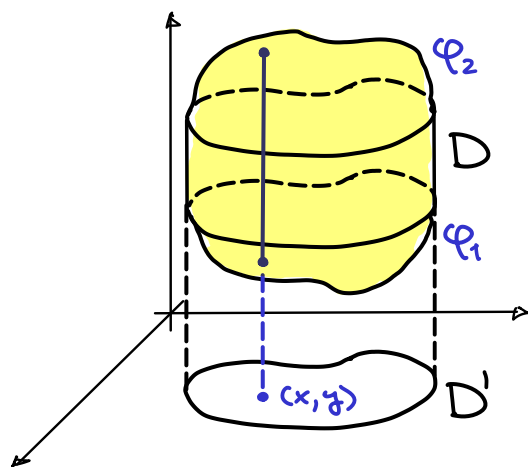
1) INTEGRAZIONE PER FILI

Se $\exists$ inv. misurabile $D' \subseteq \mathbb{R}^2$ e $\exists \varphi_1, \varphi_2 \in C(D')$ tali che

$$D = \{ (x,y,z) : (x,y) \in D', z \in [\varphi_1(x,y), \varphi_2(x,y)] \}$$

allora

$$\iiint_D f(x,y,z) dx dy dz = \iint_{D'} \left(\int_{z=\varphi_1(x,y)}^{\varphi_2(x,y)} f(x,y,z) dz \right) dx dy$$



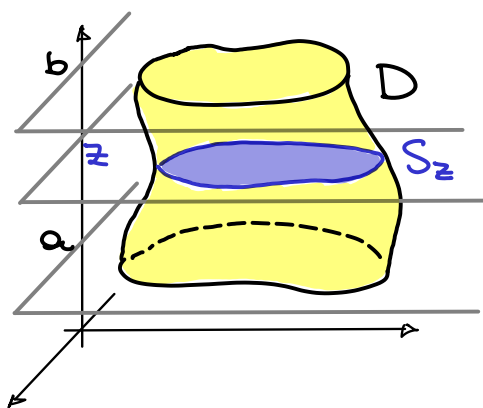
2) INTEGRAZIONE PER SEZIONI

Se $\forall z \in [a, b] \exists$ inv. misurabile $S_z \subseteq \mathbb{R}^2$ tale che

$$D = \{(x, y, z) : z \in [a, b] \text{ e } (x, y) \in S_z\}$$

allora

$$\iiint_D f(x, y, z) dx dy dz = \int_{z=a}^b \left(\iint_{S_z} f(x, y, z) dx dy \right) dz$$



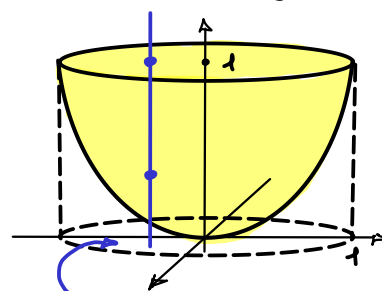
ESEMPI

- $\iiint_D |x| z dx dy dz \quad D = \{(x, y, z) : x^2 + y^2 \leq z \leq 1\}.$

1) Integrazione per fili

$$= \iint_{D'} |x| \left(\int_{z=x^2+y^2}^1 z dz \right) dx dy$$

$$= \iint_{D'} |x| \cdot \left[\frac{z^2}{2} \right]_{x^2+y^2}^1 dx dy = \frac{1}{2} \iint_{D'} |x| (1 - (x^2 + y^2)^2) dx dy$$



$$D' = \{(x, y) : x^2 + y^2 \leq 1\}$$

$$\stackrel{CP_1}{=} \frac{1}{2} \int_{\rho=0}^1 \int_{\theta=0}^{2\pi} \rho |\cos \theta| \cdot (1 - \rho^4) \rho d\rho d\theta$$

$$\int_0^{2\pi} |\cos \theta| d\theta = 4 \int_0^{\pi/2} \cos \theta d\theta = 4 [\sin \theta]_0^{\pi/2} = 4$$

$$= \frac{4}{2} \int_0^1 \rho^2 (1 - \rho^4) d\rho = 2 \left[\frac{\rho^3}{3} - \frac{\rho^5}{5} \right]_0^1 = \frac{2(7-3)}{15} = \frac{8}{15}.$$

2) Integrazione per sezioni

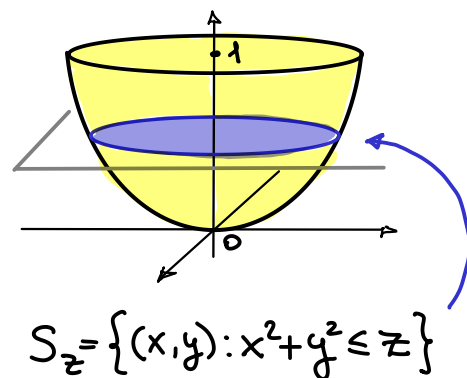
$$= \int_{z=0}^1 z \left(\iint_{S_z} |x| dx dy \right) dz$$

c.p.

$$= \int_{z=0}^1 z \left(\int_{\rho=0}^{\sqrt{z}} \int_{\theta=0}^{2\pi} \rho |\cos \theta| \cdot \rho d\rho d\theta \right) dz$$

$$= \int_{z=0}^1 z \left(\int_{\rho=0}^{\sqrt{z}} 4\rho^2 d\rho \right) dz = 4 \int_{z=0}^1 z \left[\frac{\rho^3}{3} \right]_0^{\sqrt{z}} dz = \frac{4}{3} \int_0^1 z^{1+3/2} dz$$

$$= \frac{4}{3} \left[\frac{2}{7} z^{7/2} \right]_0^1 = \frac{8}{21}.$$



• Calcolare il volume dell'insieme

$$D = \{(x,y,z): x^2 + y^2 \leq 1, x^2 + z^2 \leq 1\}.$$

D è l'intersezione di due cilindri.

Sia $E = D \cap \{(x,y,z): x,y,z \geq 0\}$

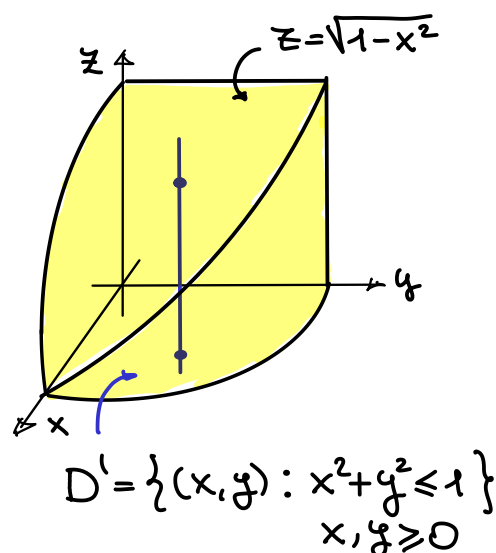
Per simmetria $|D| = 8|E|$.

1) Integrazione per folie

$$|D| = 8|E| = 8 \iint_{D'} \left(\int_{z=0}^{\sqrt{1-x^2}} 1 dz \right) dx dy$$

$$= 8 \iint_{D'} (\sqrt{1-x^2}) dx dy = 8 \int_{x=0}^1 \sqrt{1-x^2} \left(\int_{y=0}^{\sqrt{1-x^2}} 1 dy \right) dx$$

$$= 8 \int_0^1 (1-x^2) dx = 8 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{16}{3}.$$

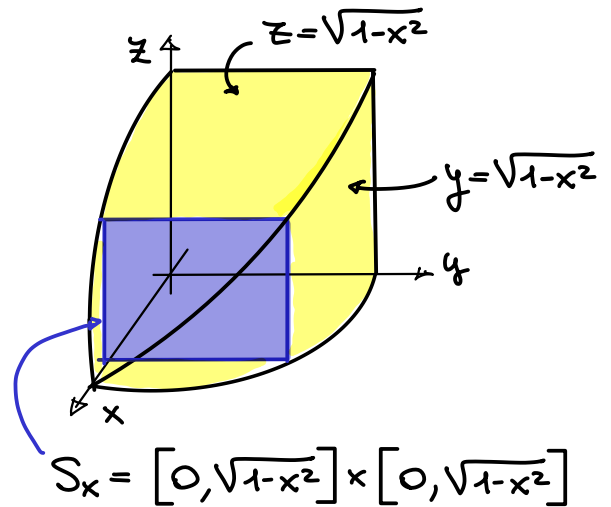


2) Integrazione per sezioni lungo l'asse x

$$|D| = 8 \int_{x=0}^1 \left(\iint_{S_x} 1 \, dy \, dz \right) dx$$

$$= 8 \int_{x=0}^1 |S_x| \, dx$$

$$= 8 \int_0^1 (\sqrt{1-x^2})^2 \, dx = 8 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{16}{3}.$$



TEOREMA (CAMBIO DI VARIABILI PER INTEGRALI TRIPLI)

Sia $\vec{\Phi}: A \rightarrow \vec{\Phi}(A)$ una funzione biunivoca con A aperto di $\mathbb{R}^3$ e $\vec{\Phi} \in C^1(A)$. Sia $D \subseteq \vec{\Phi}(A)$ un insieme limitato e misurabile e sia f una funzione continua e limitata in D . Se $\det(J_{\vec{\Phi}}(u,v,w)) \neq 0$ in $\vec{\Phi}^{-1}(D) \setminus E$ con $|E|=0$ allora

$$\iiint_D f(x,y,z) \, dx \, dy \, dz = \iiint_{\vec{\Phi}^{-1}(D)} f(\vec{\Phi}(u,v,w)) |\det(J_{\vec{\Phi}}(u,v,w))| \, du \, dv \, dw$$

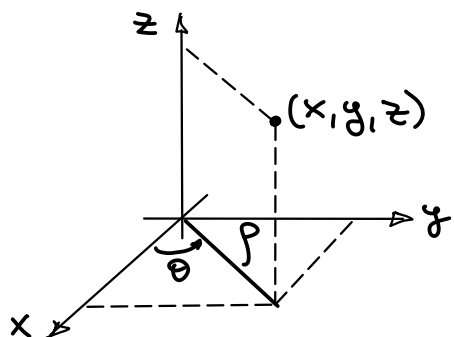
$\begin{cases} x = \varphi_1(u,v,w) \\ y = \varphi_2(u,v,w) \\ z = \varphi_3(u,v,w) \end{cases} \quad \vec{\Phi}(u,v,w)$

dove la MATRICE JACOBIANA di $\vec{\Phi}$ è

$$J_{\vec{\Phi}}(u,v,w) = \begin{bmatrix} \frac{\partial \varphi_1}{\partial u}(u,v,w) & \frac{\partial \varphi_1}{\partial v}(u,v,w) & \frac{\partial \varphi_1}{\partial w}(u,v,w) \\ \frac{\partial \varphi_2}{\partial u}(u,v,w) & \frac{\partial \varphi_2}{\partial v}(u,v,w) & \frac{\partial \varphi_2}{\partial w}(u,v,w) \\ \frac{\partial \varphi_3}{\partial u}(u,v,w) & \frac{\partial \varphi_3}{\partial v}(u,v,w) & \frac{\partial \varphi_3}{\partial w}(u,v,w) \end{bmatrix}$$

Vediamo il calcolo di $\det(J_{\Phi})$ per due cambi di coordinate in dimensione tre.

1) COORDINATE CILINDRICHE (ρ, θ, z) :



$$\begin{cases} x = \varphi_1(\rho, \theta, z) = \rho \cos \theta \\ y = \varphi_2(\rho, \theta, z) = \rho \sin \theta \\ z = \varphi_3(\rho, \theta, z) = z \end{cases}$$

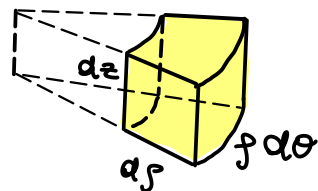
$$\rho = \sqrt{x^2 + y^2}$$

$$\theta \in [0, 2\pi)$$

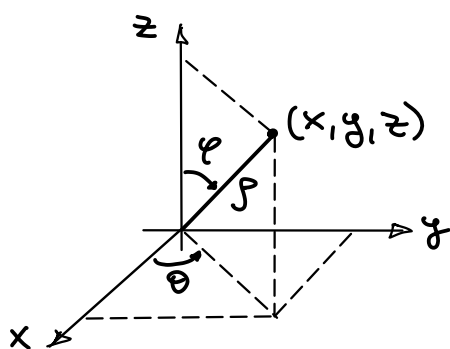
$$J_{\Phi}(\rho, \theta, z) = \begin{bmatrix} \cos \theta & -\rho \sin \theta & 0 \\ \sin \theta & \rho \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

e dunque $|\det(J_{\Phi}(\rho, \theta, z))| = \rho$.

L'elemento infinitesimo di volume è $\rho d\rho d\theta dz$



2) COORDINATE SFERICHE (ρ, θ, φ) :



$$\begin{cases} x = \varphi_1(\rho, \theta, \varphi) = \rho \sin \varphi \cos \theta \\ y = \varphi_2(\rho, \theta, \varphi) = \rho \sin \varphi \sin \theta \\ z = \varphi_3(\rho, \theta, \varphi) = \rho \cos \varphi \end{cases}$$

$$\rho = \sqrt{x^2 + y^2 + z^2}$$

$$\theta \in [0, 2\pi)$$

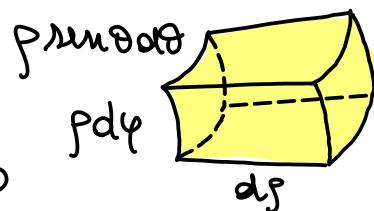
$$\varphi \in [0, \pi]$$

$$J_{\Phi}(\rho, \theta, \varphi) = \begin{bmatrix} \sin \varphi \cos \theta & -\rho \sin \varphi \sin \theta & \rho \cos \varphi \cos \theta \\ \sin \varphi \sin \theta & \rho \sin \varphi \cos \theta & \rho \cos \varphi \sin \theta \\ \cos \varphi & 0 & -\rho \sin \varphi \end{bmatrix}$$

e dunque $|\det(J_{\Phi}(\rho, \theta, \varphi))| = \rho^2 \sin \varphi$

L'elemento infinitesimo di volume è

$$\rho^2 \sin \varphi d\rho d\theta d\varphi$$

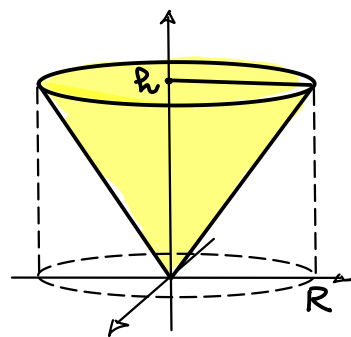


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- Calcolare il volume dell'insieme

$$D = \left\{ (x, y, z) : \frac{x^2 + y^2}{R^2} \leq \frac{z^2}{h^2}, 0 \leq z \leq h \right\}$$

↑
doppio come infinito



Φ coordinate
cilindriche

$$\Phi^{-1}(D) = \left\{ (p, \theta, z) : p \in [0, R], \frac{h}{R}p \leq z \leq h, \theta \in [0, 2\pi) \right\}$$

$$|D| = \iiint_D 1 \, dx \, dy \, dz = \iiint_{\Phi^{-1}(D)} 1 \, p \, dp \, d\theta \, dz$$

$$= \iint_{[0, R] \times [0, 2\pi)} \left(\int_{z = \frac{h}{R}p}^h dz \right) p \, dp \, d\theta = \int_{p=0}^R \left(\int_{\theta=0}^{2\pi} d\theta \right) \left(h - \frac{h}{R}p \right) p \, dp$$

$$= 2\pi h \int_0^R \left(p - \frac{1}{R}p^2 \right) dp = 2\pi h \left[\frac{p^2}{2} - \frac{1}{R} \cdot \frac{p^3}{3} \right]_0^R = 2\pi h \frac{R^2}{6} = \frac{\pi R^2 h}{3}$$

- $\iiint_D z \, dx \, dy \, dz$ $D = \left\{ (x, y, z) : x^2 + y^2 + z^2 \leq R^2, z \geq 0 \right\}$

↑
semisfera

Φ coordinate
sferiche

$$\Phi^{-1}(D) = \left\{ (p, \theta, \varphi) \in [0, R] \times [0, 2\pi) \times [0, \frac{\pi}{2}] \right\}$$

$$\iiint_D z \, dx \, dy \, dz = \iiint_{\Phi^{-1}(D)} p \cos \varphi \, p^2 \sin \varphi \, dp \, d\theta \, d\varphi$$

$$= \int_0^R p^3 \, dp \int_0^{2\pi} d\theta \int_0^{\frac{\pi}{2}} \cos \varphi \sin \varphi \, d\varphi = 2\pi \left[\frac{p^4}{4} \right]_0^R \cdot \left[\frac{\sin^2 \varphi}{2} \right]_0^{\frac{\pi}{2}}$$

sin φ d(sin φ)

$$= 2\pi \cdot \frac{R^4}{4} \cdot \frac{1}{2} = \frac{\pi}{4} R^4$$