

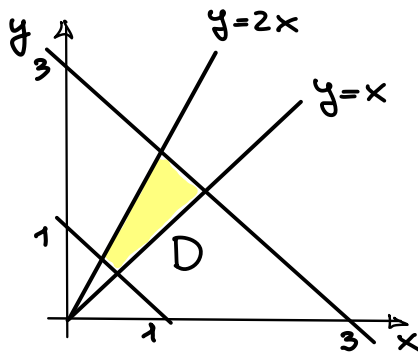
ANALISI MATEMATICA 2 - LEZIONE 19

ESEMPI

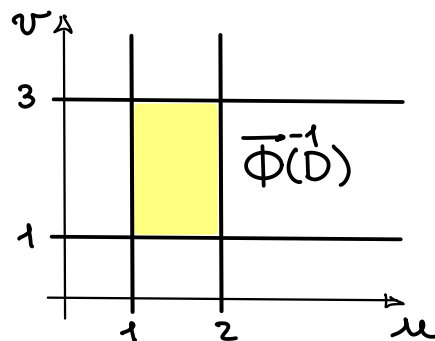
$$\bullet \iint_D \frac{1}{xy} dx dy \quad D = \{(x, y) : 0 \leq x \leq y \leq 2x, 1 \leq x+y \leq 3\}.$$

Poniamo

$$\begin{cases} u = \frac{y}{x} \\ v = x+y \end{cases} \quad \vec{\Phi}^{-1}(x, y) \quad \text{da cui} \quad \begin{cases} x = \frac{v}{u+1} \\ y = \frac{uv}{u+1} \end{cases} \quad \vec{\Phi}(u, v)$$



$\vec{\Phi}^{-1}$
 $\vec{\Phi}$



Inoltre

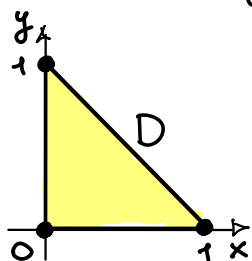
$$\det(J_{\vec{\Phi}}(u, v)) = \det \begin{pmatrix} -\frac{v}{(u+1)^2} & \frac{1}{u+1} \\ \frac{v}{(u+1)^2} & \frac{u}{u+1} \end{pmatrix} = \frac{-vu - v}{(u+1)^3} = \frac{-v}{(u+1)^2}$$

Allora

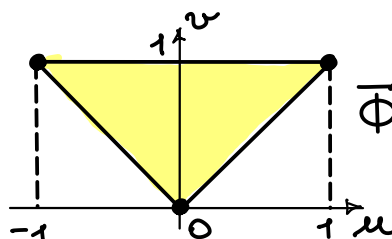
$$\begin{aligned} \iint_D \frac{1}{xy} dx dy &= \iint_{\vec{\Phi}^{-1}(D)} \frac{u+1}{v} \cdot \frac{u+1}{uv} \cdot \left| \frac{-v}{(u+1)^2} \right| du dv \\ &= \int_{u=1}^2 \left(\int_{v=1}^3 \frac{(u+1)^2}{u v^2} \cdot \frac{v}{(u+1)^2} dv \right) du \\ &= \int_1^2 \frac{1}{u} \left[\log(v) \right]_1^3 du \\ &= \log(3) \cdot \left[\log(u) \right]_1^2 = \log(3) \cdot \log(2). \end{aligned}$$

$$\bullet \iint_D \frac{(x-y)^2}{1+x+y} dx dy \quad D = \{(x,y) : x \geq 0, y \geq 0, x+y \leq 1\}.$$

Poniamo $\begin{cases} u = x-y \\ v = x+y \end{cases}$ $\vec{\Phi}^{-1}(x,y)$ da cui $\begin{cases} x = \frac{u+v}{2} \\ y = \frac{v-u}{2} \end{cases}$ $\vec{\Phi}(u,v)$



$\vec{\Phi}^{-1}$
 $\vec{\Phi}$



$\vec{\Phi}^{-1}(D)$

$\vec{\Phi}$
 $(0,0) \xrightarrow{\vec{\Phi}} (0,0)$
 $(1,1) \rightarrow (1,0)$
 $(-1,1) \rightarrow (0,1)$
 $\vec{\Phi}$ è lineare

Inoltre

$$\det(J_{\vec{\Phi}}(u,v)) = \det\left(\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\right) = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Allora

$$\iint_D \frac{(x-y)^2}{1+x+y} dx dy = \iint_{\vec{\Phi}^{-1}(D)} \frac{u^2}{1+v} \left|\frac{1}{2}\right| du dv$$

u -semplice $\rightarrow$

$$= \frac{1}{2} \int_{v=0}^1 \left(\int_{u=-v}^v \frac{u^2}{1+v} du \right) dv$$

$$= \frac{1}{2} \int_{v=0}^1 \frac{1}{1+v} \left[\frac{u^3}{3} \right]_{-v}^v dv$$

$$= \frac{1}{3} \int_0^1 \frac{v^3}{1+v} dv$$

$$= \frac{1}{3} \int_0^1 \left(1 - v + v^2 - \frac{1}{1+v} \right) dv$$

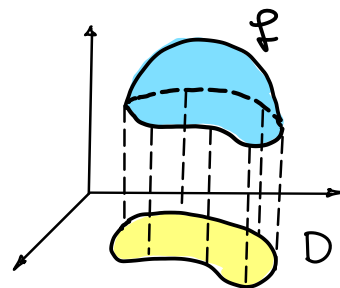
$$= \frac{1}{3} \left[v - \frac{v^2}{2} + \frac{v^3}{3} - \log(1+v) \right]_0^1$$

$$= \frac{1}{3} \left(1 - \frac{1}{2} + \frac{1}{3} - \log(2) \right) = \frac{5}{18} - \frac{\log(2)}{3}.$$

OSSERVAZIONE

Se $f(x,y) \geq 0$ per ogni $(x,y) \in D$ allora

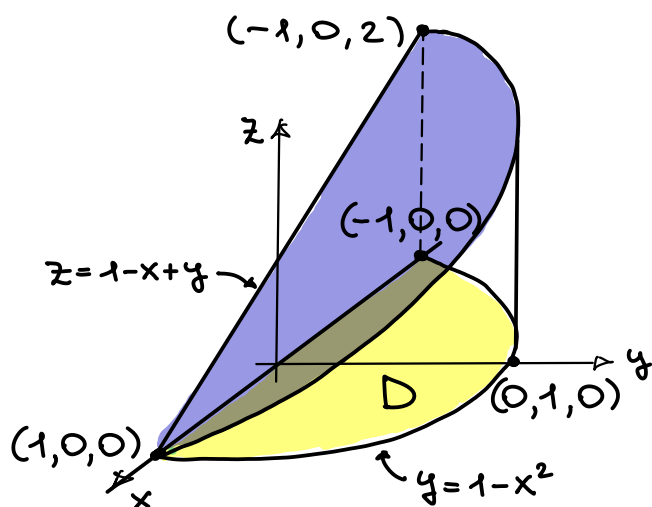
$$\iint_D f(x,y) dx dy = \text{Volume}(E) = |E|$$



dove $E = \{(x,y,z) : (x,y) \in D, 0 \leq z \leq f(x,y)\}$.

- Calcolare il volume di

$$E = \{(x,y,z) : y \geq 0, z \geq 0, x - y + z \leq 1, y \leq 1 - x^2\}$$



Consideriamo

$$D = \{(x,y) : 0 \leq y \leq 1 - x^2\}$$

e

$$z = f(x,y) = 1 - x + y.$$

Si noti che se $(x,y) \in D$ allora

$$f(x,y) \geq 1 - x \geq 0.$$

$\uparrow$ $y \geq 0$ $\uparrow$ $x \leq 1$

$$|E| = \iiint_D (1 - x + y) dx dy = \int_{x=-1}^1 \left(\int_{y=0}^{1-x^2} (1 - x + y) dy \right) dx$$

$$= \int_{-1}^1 \left[y - xy + \frac{y^2}{2} \right]_0^{1-x^2} dx = \int_{-1}^1 \left((1-x^2) - x(1-x^2) + \frac{(1-x^2)^2}{2} \right) dx$$

$\swarrow$ $[-1,1]$ è simmetrico
pari dispari pari

$$= \int_0^1 (2(1-x^2) + (1-2x^2+x^4)) dx = \int_0^1 (3-4x^2+x^4) dx$$

$$= \left[3x - \frac{4x^3}{3} + \frac{x^5}{5} \right]_0^1 = \frac{45-20+3}{15} = \frac{28}{15}.$$

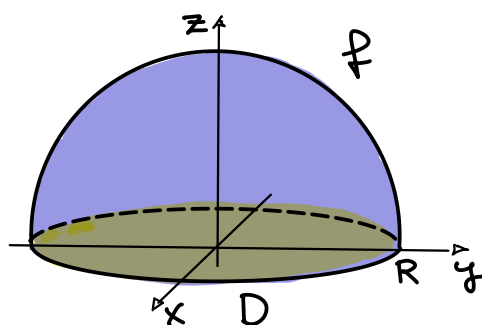
- Calcolare il volume di una sfera di raggio R .

La sfera di raggio R e centro $(0,0,0)$ ha equazione

$$x^2 + y^2 + z^2 = R^2.$$

Sia $E = \{(x,y,z) : x^2 + y^2 + z^2 \leq R^2\}$ e consideriamo la semisfera data dal grafico della funzione

$$z = f(x,y) = \sqrt{R^2 - x^2 - y^2} \quad \text{in } D = \{(x,y) : x^2 + y^2 \leq R^2\}.$$



Allora

$$|E| = 2 \iint_D f(x,y) \, dx \, dy$$

$$= 2 \iint_{\{x^2 + y^2 \leq R^2\}} \sqrt{R^2 - x^2 - y^2} \, dx \, dy$$

$$\stackrel{CP}{=} 2 \iint_{[0,R] \times [0,2\pi]} \sqrt{R^2 - \rho^2} \, \rho \, d\rho \, d\theta$$

$$= 2 \int_{\rho=0}^R \sqrt{R^2 - \rho^2} \left(\int_{\theta=0}^{2\pi} d\theta \right) \rho \, d\rho$$

$$= 4\pi \int_0^R \sqrt{R^2 - \rho^2} \, \rho \, d\rho$$

$$\begin{aligned} & \begin{matrix} t = R^2 - \rho^2 \\ dt = -2\rho d\rho \end{matrix} \quad \rightarrow \quad \cancel{4}\pi \int_{R^2}^0 \sqrt{t} \left(\frac{dt}{\cancel{-2}} \right) = 2\pi \left[\frac{2t^{3/2}}{3} \right]_0^{R^2} = \frac{4\pi}{3} R^3. \end{aligned}$$

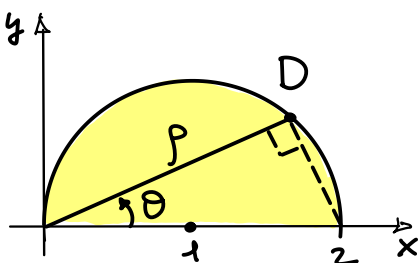
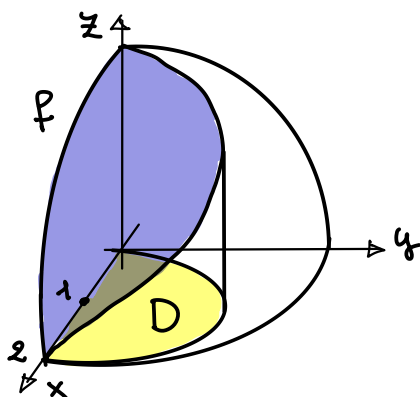
- Calcolare il volume dell'insieme

$$E = \{(x, y, z) : x, y, z \geq 0, x^2 + y^2 + z^2 \leq 4, x^2 + y^2 \leq 2x\}$$

sfera

$$(x-1)^2 + y^2 \leq 1$$

cilindro



Allora

Consideriamo la funzione

$$z = f(x, y) = \sqrt{4 - x^2 - y^2}$$

$$\text{in } D = \{(x, y) : x^2 + y^2 \leq 2x, y \geq 0\}.$$

In coordinate polari

$$\vec{\Phi}^{-1}(D) = \{(r, \theta) : 0 \leq r \leq 2\cos\theta, \theta \in [0, \frac{\pi}{2}]\}$$

$$\begin{aligned} |E| &= \iint_D \sqrt{4 - x^2 - y^2} \, dx \, dy \stackrel{CP}{=} \iint_{\vec{\Phi}^{-1}(D)} \sqrt{4 - r^2} \cdot r \, dr \, d\theta \\ &= \int_{\theta=0}^{\pi/2} \left(\int_{r=0}^{2\cos\theta} \sqrt{4 - r^2} \cdot r \, dr \right) d\theta \\ &\stackrel{t=4-r^2}{dt=-2rdr} = \int_0^{\pi/2} \left(\int_{t=4}^{0} \sqrt{t} \left(-\frac{dt}{2}\right) \right) d\theta = \int_0^{\pi/2} \left[\frac{t^{3/2}}{3} \right]_{4\sin^2\theta}^4 d\theta \\ &= \frac{8}{3} \int_0^{\pi/2} (1 - \sin^3\theta) d\theta = \\ &= \frac{8}{3} \cdot \frac{\pi}{2} + \frac{8}{3} \int_0^{\pi/2} (1 - \cos^2\theta) d(\cos\theta) \\ &= \frac{4\pi}{3} + \frac{8}{3} \left[\cos\theta - \frac{\cos^3\theta}{3} \right]_0^{\pi/2} = \frac{4\pi}{3} - \frac{16}{9}. \end{aligned}$$

• Calcolare

$$1) \lim_{n \rightarrow \infty} \iint_{D_n} e^{-(x^2+y^2)} dx dy \quad \text{con } D_n = \{(x,y) : x^2+y^2 \leq n^2\}.$$

$$2) \lim_{n \rightarrow \infty} \iint_{Q_n} e^{-(x^2+y^2)} dx dy \quad \text{con } Q_n = [-n, n] \times [-n, n].$$

$$1) \iint_{D_n} e^{-(x^2+y^2)} dx dy \stackrel{CP}{=} \int_{\rho=0}^n e^{-\rho^2} \left(\int_{\theta=0}^{2\pi} d\theta \right) \rho d\rho$$

$$= 2\pi \int_0^n e^{-\rho^2} \rho d\rho = \cancel{2}\pi \left[-\frac{e^{-\rho^2}}{\cancel{2}} \right]_0^n = \pi(1 - e^{-n^2}) \xrightarrow{n \rightarrow \infty} \pi.$$

$$2) \iint_{Q_n} e^{-(x^2+y^2)} dx dy = \int_{x=-n}^n e^{-x^2} \left(\int_{y=-n}^n e^{-y^2} dy \right) dx$$

← non dipende da x

$$= \left(\int_{-n}^n e^{-x^2} dx \right) \cdot \left(\int_{-n}^n e^{-y^2} dy \right) \xrightarrow{n \rightarrow \infty} \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2.$$

OSSERVAZIONE

I due limiti ottenuti in 1) e 2) sono uguali? SÌ

$$0 \leq \iint_{Q_n \setminus D_n} e^{-(x^2+y^2)} dx dy = \iint_{Q_n \setminus D_n} e^{-(x^2+y^2)} dx dy$$

↑
 $Q_n \supseteq D_n$
e $\emptyset \neq Q_n \setminus D_n$



$\leq \sup_{Q_n \setminus D_n} e^{-(x^2+y^2)} \cdot |Q_n \setminus D_n| = e^{-n^2} \cdot (4n^2 - \pi n^2)$

→ 0 per $n \rightarrow \infty$

Quindi $\pi = \left(\int_{-\infty}^{+\infty} e^{-x^2} dx \right)^2$ ossia $\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}$ che NON si può ottenere integrando direttamente poiché le primitive di e^{-x^2} non sono esprimibili mediante funzioni elementari.