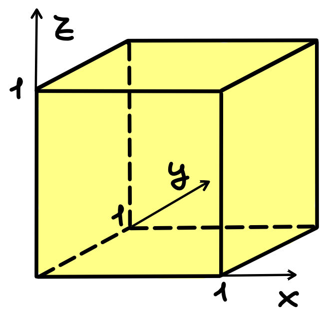
ANALISI MATEMATICA 2 - FOGLIO 8

1.a Calcolare $\iint_S (x^2+z^2)\,dS$ dove S è la superficie del cubo $[0,1]^3$.



Per simmetria $\iint_S x^2 dS = \iint_S z^2 dS$.

$S=\bigcup_{i=1}^{6} S_i$ è regolare a pezzi:

$S_1=\{(x,y,0): x,y\in[0,1]\}$, $S_2=\{(x,y,1): x,y\in[0,1]\}$

e quindi

$$\iint_{S_1} x^2 dS = \iint_{S_2} x^2 dS = \int_0^1\int_0^1 x^2\,dx\,dy = \left[\frac{x^3}{3}\right]_0^1 \left[y\right]_0^1 = \frac{1}{3}.$$

Inoltre

$S_3=\{(x,0,z): x,z\in[0,1]\}$, $S_4=\{(x,1,z): x,z\in[0,1]\}$,

$S_5=\{(0,y,z): y,z\in[0,1]\}$, $S_6=\{(1,y,z): y,z\in[0,1]\}$

e così

$$\iint_{S_3} x^2 dS = \iint_{S_4} x^2 dS = \int_0^1\int_0^1 x^2\,dx\,dz = \left[\frac{x^3}{3}\right]_0^1 \left[z\right]_0^1 = \frac{1}{3},$$

$$\iint_{S_5} \underset{0}{\underset{\shortparallel}{x^2}}\, dS = 0, \quad \iint_{S_6} \underset{1}{\underset{\shortparallel}{x^2}}\, dS = |S_6| = 1.$$

Infine

$$\iint_S (x^2+z^2)\,dS = 2\sum_{i=1}^{6}\iint_{S_i} x^2 dS = 2\left(4\cdot\frac{1}{3}+0+1\right) = \frac{14}{3}.$$

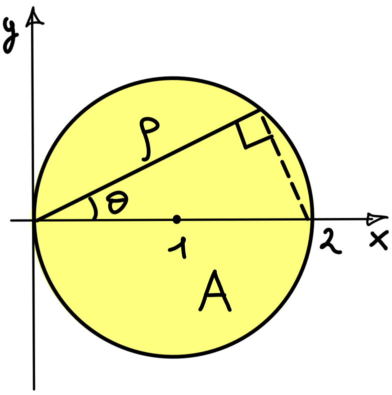
$\boxed{1.b}$ Calcolare $\iint_S x\sqrt{z^2-4x^2}\,dS$ dove S è

$$S=\{(x,y,z):\ \underbrace{z=2\sqrt{x^2+y^2}}_{\text{cono}},\ \underbrace{(x-1)^2+y^2\le 1}_{\text{cilindro}}\}.$$

Parametrizzazione di S:

$$\vec{\sigma}(x,y)=\left(x,y,2\sqrt{x^2+y^2}\right)$$

con $A=\{(x,y):(x-1)^2+y^2\le 1\}$.



Così

$$\vec{\sigma_x}\times\vec{\sigma_y}=\left(-\frac{2x}{\sqrt{x^2+y^2}},-\frac{2y}{\sqrt{x^2+y^2}},1\right).$$

Quindi $\|\vec{\sigma_x}\times\vec{\sigma_y}\|=\sqrt{5}$ e

$$\iint_S x\sqrt{z^2-4x^2}\,dS=\iint_A x\sqrt{\overbrace{\cancel{4x^2}+4y^2-\cancel{4x^2}}^{=2|y|}}\cdot\sqrt{5}\,dx\,dy$$

$$\overset{CP}{=}2\sqrt{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{2\cos\theta}\rho\cos\theta\cdot\rho|\sin\theta|\,\rho\,d\rho\,d\theta$$

$$\overset{CP}{=}2\sqrt{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta\cdot|\sin\theta|\left[\frac{\rho^4}{4}\right]_0^{2\cos\theta}d\theta$$

$$=8\sqrt{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\overset{\theta\text{-pari}}{\cos^5\theta\cdot|\sin\theta|}\,d\theta=16\sqrt{5}\int_0^{\frac{\pi}{2}}\cos^5\theta\,d(-\cos\theta)$$

$$=16\sqrt{5}\left[-\frac{\cos^6\theta}{6}\right]_0^{\frac{\pi}{2}}=\frac{8\sqrt{5}}{3}.$$

$\boxed{1.c}$ Calcolare $\iint_S \frac{z^2}{(1+x^2+y^2)^{3/2}}\,dS$ dove S è

$S=\{(r\cos t, r\,\mathrm{sen}\,t, t) : r\in[0,1], t\in[0,2\pi]\}$. elicoide

Parametrizzazione di S:

$\vec{\sigma}(r,t)=(r\cos t, r\,\mathrm{sen}\,t, t)$ con $A=[0,1]\times[0,2\pi]$.

Così

$$\vec{\sigma_r}\times\vec{\sigma_t}=\det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \cos t & \mathrm{sen}\,t & 0\\ -r\,\mathrm{sen}\,t & r\cos t & 1\end{bmatrix}=(\mathrm{sen}\,t,-\cos t, r).$$

Quindi $\|\vec{\sigma_r}\times\vec{\sigma_t}\|=\sqrt{1+r^2}$ e

$$\iint_S \frac{z^2}{(1+x^2+y^2)^{3/2}}\,dS=\iint_A \frac{t^2}{(1+r^2)^{3/2}}\sqrt{1+r^2}\,dr\,dt$$

$$=\int_0^{2\pi} t^2\,dt\cdot\int_0^1 \frac{dr}{1+r^2}$$

$$=\left[\frac{t^3}{3}\right]_0^{2\pi}\cdot\left[\mathrm{arctg}(r)\right]_0^1$$

$$=\frac{8\pi^3}{3}\cdot\frac{\pi}{4}=\frac{2\pi^4}{3}$$

$\boxed{1.d}$ Calcolare $\iint_S \sqrt{(1+x)(1+y)-1}\,dS$ dove S è

$S=\{(u^2,v^2,2(u-v)), u,v\in[0,1]\}$. grafico di $f(x,y)=2(\sqrt{x}-\sqrt{y})$ per $(x,y)\in[0,1]^2$.

Parametrizzazione di S:

$\vec{\sigma}(u,v)=(u^2,v^2,2(u-v))$ con $A=[0,1]^2$.

Così

$$\vec{\sigma_u} \times \vec{\sigma_v} = \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2u & 0 & 2 \\ 0 & 2v & -2 \end{bmatrix} = (-4v,\ 4u,\ 4uv).$$

Quindi $\|\vec{\sigma_u} \times \vec{\sigma_v}\| = 4\sqrt{v^2+u^2+u^2v^2}$ e

$$\iint_S \sqrt{\underbrace{(1+x)(1+y)-1}_{x+y+xy}}\, dS = \iint_A \sqrt{u^2+v^2+u^2v^2} \cdot 4\sqrt{v^2+u^2+u^2v^2}\, du\, dv$$

$$= 4\int_0^1\!\!\int_0^1 (u^2+v^2+u^2v^2)\, du\, dv$$

$$= 4\left(\left[\frac{u^3}{3}\right]_0^1 \cdot 1 + \left[\frac{v^3}{3}\right]_0^1 \cdot 1 + \left[\frac{u^3}{3}\right]_0^1 \cdot \left[\frac{v^3}{3}\right]_0^1 \right)$$

$$= \frac{4}{3}\left(2+\frac{1}{3}\right) = \frac{28}{9}.$$

$\boxed{2.a}$ Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove

$\vec{F} = (3e^{x+y}, 2y-z, xy)$, $S = \partial D$ e

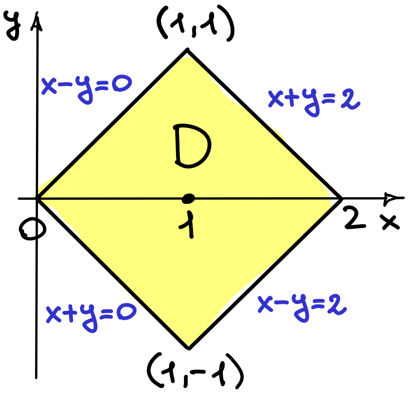
$$D = \{(x,y,z) : |x-1| + |y| \leq 1,\ 0 \leq z \leq x+y\}.$$

S è orientata verso l'esterno.

Applichiamo il teorema della divergenza:

$$\mathrm{div}(\vec{F}) = \frac{\partial}{\partial x}(3e^{x+y}) + \frac{\partial}{\partial y}(2y-z) + \frac{\partial}{\partial z}(xy) = 3e^{x+y} + 2$$

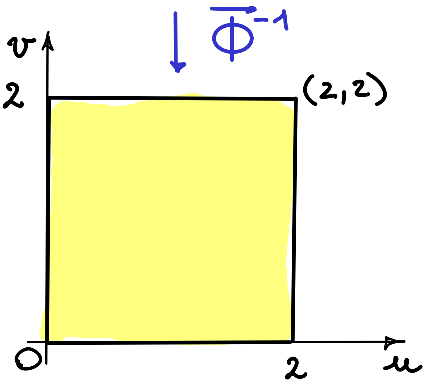
e allora



$$\iint_S \langle \vec{F}, d\vec{S} \rangle \overset{TD}{=} \iiint_D \mathrm{div}(\vec{F})\,dx\,dy\,dz$$

$$= \iint_{\{|x-1|+|y|\leq 1\}} (3e^{x+y}+2)\left(\int_{z=0}^{x+y} dz\right) dx\,dy$$

$$= \iint_{\{|x-1|+|y|\leq 1\}} (3e^{x+y}+2)(x+y)\,dx\,dy$$



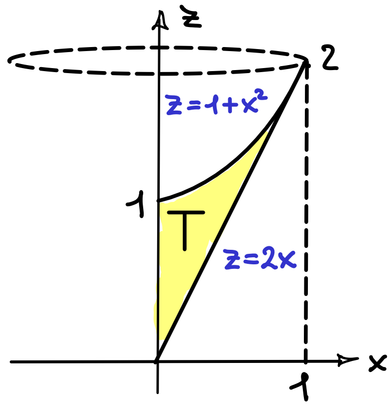
$$\begin{cases} u = x+y \\ v = x-y \end{cases} \quad \vec{\Phi}^{-1}$$

$$\overset{\mathcal{J}}{=} \iint_{[0,2]^2 = \vec{\Phi}^{-1}(\{|x-1|+|y|\leq 1\})} (3e^u+2)\,u \underbrace{\left(\frac{1}{2}\right)}_{= |\det J_{\vec{\Phi}}|} du\,dv$$

$$= \int_0^2 (3e^u u + 2u)\,du = \left[3e^u(u-1) + u^2\right]_0^2$$

$$= 3e^2 + 4 - (-3) = 3e^2 + 7.$$

2.b Calcolare il flusso $\iint_S \langle \vec F, d\vec S\rangle$ dove $\vec F = (e^{y^2}, 5x^2y, 5y^2z)$, $S = \partial D$ è orientata verso l'esterno e D è generato dalla rotazione completa di $T = \{(x,0,z) : 0 \le 2x \le z \le 1+x^2 \le 2\}$ intorno all'asse z.



Applichiamo il teorema della divergenza:

$$\mathrm{div}(\vec F) = \frac{\partial}{\partial x}(e^{y^2}) + \frac{\partial}{\partial y}(5x^2y) + \frac{\partial}{\partial z}(5y^2z) = 5(x^2+y^2).$$

e allora

$$\iint_S \langle \vec F, d\vec S\rangle \overset{TD}{=} \iiint_D \mathrm{div}(\vec F)\,dx\,dy\,dz$$

$$\overset{CC}{=} \int_{\rho=0}^{1} 5\rho^2 \left(\int_{z=2\rho}^{1+\rho^2} \left(\int_{\theta=0}^{2\pi} d\theta\right) dz\right) \rho\, d\rho = 10\pi \int_0^1 \rho^3 (1+\rho^2-2\rho)\, d\rho$$

$$= 10\pi \left[\frac{\rho^4}{4} + \frac{\rho^6}{6} - 2\frac{\rho^5}{5}\right]_0^1 = 10\pi \cdot \frac{15+10-24}{60} = \frac{\pi}{6}.$$

2.C Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove $\vec{F} = (-5x^3, 2y^2+z, \sin(xy))$ e $S = \partial D$ orientata verso l'interno con

$$D = \left\{(x,y,z) : \frac{x^2}{4} + y^2 + \frac{z^2}{2} \le 1\right\}.$$ ellissoide

Applichiamo il teorema della divergenza:

$$\mathrm{div}(\vec{F}) = \frac{\partial}{\partial x}(-5x^3) + \frac{\partial}{\partial y}(2y^2+z) + \frac{\partial}{\partial z}(\sin(xy)) = -15x^2 + 4y.$$

e allora

verso l'interno

$$\iint_S \langle \vec{F}, d\vec{S} \rangle \overset{TD}{=} -\iiint_D \mathrm{div}(\vec{F})\,dx\,dy\,dz$$

per fili

$$= -\iint_{\left\{\frac{x^2}{4}+y^2\le 1\right\}} (-15x^2+4y)\left(\int_{z=-\sqrt{2\left(1-\frac{x^2}{4}-y^2\right)}}^{\sqrt{2\left(1-\frac{x^2}{4}-y^2\right)}} 1\,dz\right)dx\,dy$$

$$= 2\sqrt{2}\iint_{\left\{\frac{x^2}{4}+y^2\le 1\right\}} (15x^2-4y)\sqrt{1-\frac{x^2}{4}-y^2}\,dx\,dy$$

$\begin{cases} x = 2\rho\cos\theta \\ y = \rho\sin\theta \end{cases}$ $\vec{\Phi}$

θ-dispari ; $|\det J_{\vec{\Phi}}|$

$$= 2\sqrt{2}\int_{\theta=0}^{2\pi}\int_{\rho=0}^{1} (60\rho^2\cos^2\theta - \underset{\shortparallel}{4\rho\sin\theta})\sqrt{1-\rho^2}\cdot(2\rho)\,d\rho\,d\theta$$

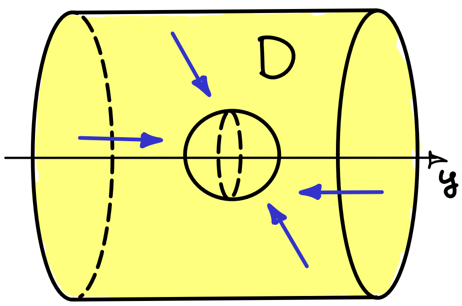
$$= 2\sqrt{2}\int_0^1 (60\rho^2\cdot\pi + 0)\sqrt{1-\rho^2}\,2\rho\,d\rho$$

$t = 1-\rho^2$
$dt = -2\rho\,d\rho$

$$= 120\sqrt{2}\pi\int_0^1 (1-t)\sqrt{t}\,dt = 120\sqrt{2}\pi\left[\frac{2t^{3/2}}{3} - \frac{2t^{5/2}}{5}\right]_0^1$$

$$= \overset{8}{\cancel{120}}\sqrt{2}\pi\cdot\frac{4}{\cancel{15}} = 32\sqrt{2}\pi.$$

2.d Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove
$\vec{F} = \frac{(x,y,z)}{(x^2+y^2+z^2)^{3/2}}$ e $S = \partial D$ orientata verso
l'interno con $D = \{(x,y,z) : x^2+z^2 \leq 4, |y| \leq 2\}$.



Dato che $\operatorname{div}(\vec{F}) = 0$ in $\mathbb{R}^3 \setminus \{(0,0,0)\}$ (visto a lezione) e D contiene il punto $(0,0,0)$, per il teorema della divergenza

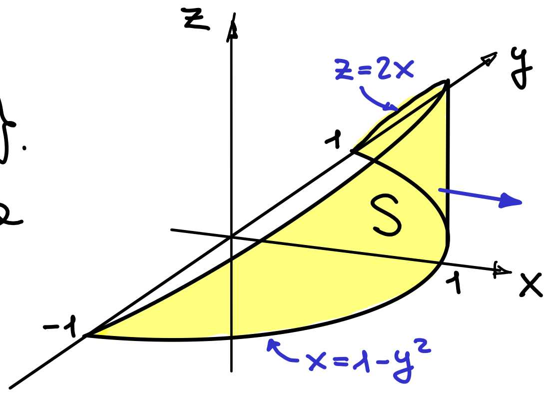
verso l'interno

$$\iint_S \langle \vec{F}, d\vec{S} \rangle = \iint_{S_2} \langle \vec{F}, d\vec{S} \rangle = \iint_{S_2} \langle \frac{\vec{n}}{r^2}, (-\vec{n}) \rangle dS$$

$$= \iint_{S_2} \frac{1}{r^2} dS = -\frac{|S_2|}{r^2} = -\frac{4\pi \cancel{r^2}}{\cancel{r^2}} = -4\pi$$

dove S_2 è una sfera di raggio $r > 0$ centrata in $(0,0,0)$ orientata verso l'interno e contenuta in D.

$\boxed{2.e}$ Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove
$\vec{F} = (5x, 2xe^{y^2}, -3z)$ e
$S = \{(x,y,z) : x+y^2=1, 0 \le z \le 2x\}$.
S è orientata in modo che
$\vec{n} = (1,0,0)$ in $(1,0,0)$.



Parametrizzazione di S:

$$\vec{\sigma}(y,z) = (1-y^2, y, z) \text{ con } A = \left\{(y,z) : \begin{array}{l} 0 \le z \le 2(1-y^2) \\ y \in [-1,1] \end{array}\right\}.$$

Così

$$\vec{\sigma}_x \times \vec{\sigma}_y = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2y & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = (1, 2y, 0)$$ compatibile con l'orientazione data

Dunque il flusso da calcolare è

$$\iint_S \langle \vec{F}, d\vec{S} \rangle = \iint_A \langle (5\overset{1-y^2}{x}, 2\overset{1-y^2}{x}e^{y^2}, -3z), (1, 2y, 0) \rangle \, dy\, dz$$

$$= \int_{y=-1}^{1} \left(\int_{z=0}^{2(1-y^2)} \left(5(1-y^2) + 4y(1-y^2)e^{y^2}\right) dz \right) dy$$

$$= \int_{y=-1}^{1} \left(\overbrace{5(1-y^2)}^{\text{pari}} + \overbrace{4y(1-y^2)e^{y^2}}^{\text{dispari}} \right) \underbrace{2(1-y^2)}_{\text{pari}} dy$$ $\leftarrow [-1,1]$ simmetrico

$$= 2 \cdot 10 \int_0^1 (1-y^2)^2 dy = 20 \left[y - \frac{2y^3}{3} + \frac{y^5}{5} \right]_0^1$$

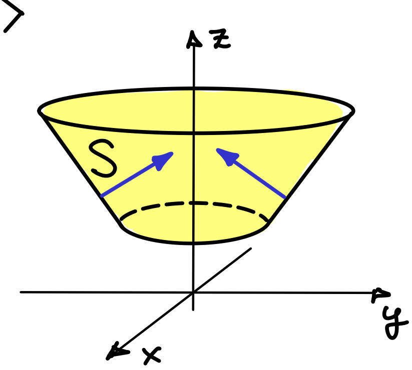
$$= 20 \cdot \frac{8}{15} = \frac{32}{3}.$$

$\boxed{2.f}$ Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$

dove $\vec{F} = (x^2, z, \log(x^2+y^2))$

e $S = \{(x,y,z) : z^2 = x^2+y^2,\ 1 \le z \le 2\}$.

S è orientata in modo che $\langle \vec{n}, \vec{k} \rangle \ge 0$.



Notiamo che S non interseca l'asse z dove $\vec{F}$ non è definito.

Parametrizzazione di S:

$$\vec{\sigma}(x,y) = (x, y, \sqrt{x^2+y^2}) \quad \text{con } A = \{(x,y) : 1 \le x^2+y^2 \le 4\}.$$

Così

$$\vec{\sigma}_x \times \vec{\sigma}_y = \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, \underbrace{1}_{>0}\right).$$

Allora

$$\iint_S \langle \vec{F}, d\vec{S} \rangle = \iint_A \left\langle (x^2, \overset{\sqrt{x^2+y^2}}{\overset{\|}{z}}, \log(x^2+y^2)), \left(-\frac{x}{\sqrt{x^2+y^2}}, -\frac{y}{\sqrt{x^2+y^2}}, 1\right) \right\rangle dx\,dy$$

$$= \iint_A \left(\overset{x\text{-dispari}}{-\frac{x^3}{\sqrt{x^2+y^2}}} \overset{y\text{-dispari}}{- y} + \log(x^2+y^2)\right) dx\,dy$$

A è simmetrico rispetto $x=0$, $y=0$

$$\overset{CP}{=} \int_{\rho=1}^{2} \int_{\theta=0}^{2\pi} \log(\rho^2)\rho\, d\rho\, d\theta = 2\pi \left[\log(\rho)\rho^2 - \frac{\rho^2}{2}\right]_1^2 = 8\pi \log 2 - 3\pi$$

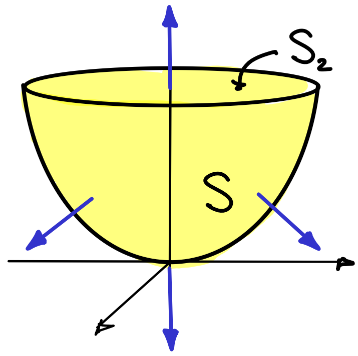
$$\int \log(\rho^2)\rho\, d\rho = \int \log(\rho) \cdot 2\rho\, d\rho = \log(\rho) \cdot \rho^2 - \int \frac{\rho^2}{\rho} d\rho = \log(\rho)\rho^2 - \frac{\rho^2}{2} + c$$

2.9 Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove $\vec{F} = (2z, 3y, x^2+4z)$ e $S = \{(x,y,z) : (x^2+y^2)^{3/2} = z \leq 1\}$

S è orientata in modo che $\vec{n} = (0,0,-1)$ in $(0,0,0)$.

In alternativa al calcolo diretto sia S_2 la superficie orientata

$$S_2 = \{(x,y,1) : x^2+y^2 \leq 1\}$$



e $\vec{n} = (0,0,1)$ e applichiamo il teorema della divergenza a

$$D = \{(x,y,z) : (x^2+y^2)^{3/2} \leq z \leq 1\}$$

in modo che $\partial^+ D = S \cup S_2$:

$$\iint_S \langle \vec{F}, d\vec{S} \rangle \overset{TD}{=} \iiint_D \operatorname{div}(\vec{F})\, dx\,dy\,dz - \iint_{S_2} \langle \vec{F}, d\vec{S} \rangle$$

$$= \frac{21\pi}{5} - \frac{17\pi}{4} = \frac{84-85}{20}\pi = -\frac{\pi}{20}.$$

Infatti

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(2z) + \frac{\partial}{\partial y}(3y) + \frac{\partial}{\partial z}(x^2+4z) = 0+3+4 = 7$$

$$\iiint_D \operatorname{div}(\vec{F})\, dx\,dy\,dz \overset{CC}{=} 7 \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \left(\int_{z=\rho^3}^{1} 1\, dz \right) \rho\, d\rho\, d\theta$$

$$= 7 \cdot 2\pi \int_0^1 (\rho - \rho^4)\, d\rho = 14\pi \left[\frac{\rho^2}{2} - \frac{\rho^5}{5} \right]_0^1 = \frac{21\pi}{5}$$

e

$$\iint_{S_2} \langle \vec{F}, \underbrace{d\vec{S}}_{(0,0,1)\,dx\,dy} \rangle = \iint_{S_2} (x^2 + 4\underbrace{z}_{=1})\, dx\,dy \overset{CP}{=} \int_{\theta=0}^{2\pi} \int_{\rho=0}^{1} \rho^2 \cos^2\theta \cdot \rho\, d\rho\, d\theta + 4|S_2|$$

$$= \pi \left[\frac{\rho^4}{4} \right]_0^1 + 4\pi = \frac{17\pi}{4}.$$

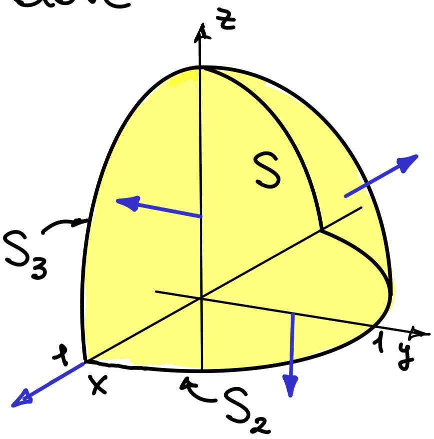
$\boxed{2.R}$ Calcolare il flusso $\iint_S \langle \vec{F}, d\vec{S} \rangle$ dove

$\vec{F} = (x^2 - y^3, 3y^2 + z, xz + y)$ e

$S = \{(x,y,z) : 0 \leq z = 1 - x^2 - y^2, \ y \geq 0\}$.

S è orientata in modo che

$\vec{n} = (0,0,1)$ in $(0,0,1)$.



In alternativa al calcolo diretto siano S_2 e S_3 le superfici orientate

$$S_2 = \{(x,y,0) : x^2 + y^2 \leq 1, \ y \geq 0\} \text{ con } \vec{n} = (0,0,-1)$$

$$S_3 = \{(x,0,z) : 0 \leq z \leq 1 - x^2\} \text{ con } \vec{n} = (0,-1,0)$$

e applichiamo il teorema della divergenza a

$$D = \{(x,y,z) : 0 \leq z \leq 1 - x^2 - y^2, \ y \geq 0\}$$

in modo che $\partial^+ D = S \cup S_2 \cup S_3$:

$$\iint_S \langle \vec{F}, d\vec{S} \rangle \overset{TD}{=} \iiint_D \operatorname{div}(\vec{F})\, dx\,dy\,dz - \iint_{S_2} \langle \vec{F}, d\vec{S} \rangle - \iint_{S_3} \langle \vec{F}, d\vec{S} \rangle$$

$$= \frac{8}{5} - \left(-\frac{2}{3} - \frac{8}{15}\right) = \frac{24 + 10 + 8}{15} = \frac{\cancel{42}^{14}}{\cancel{15}_{5}} = \frac{14}{5}.$$

Infatti:

$$\operatorname{div}(\vec{F}) = \frac{\partial}{\partial x}(x^2 - y^3) + \frac{\partial}{\partial y}(3y^2 + z) + \frac{\partial}{\partial z}(xz + y) = 3x + 6y$$

$$\iiint_D \operatorname{div}(\vec{F})\, dx\,dy\,dz \overset{CC}{=} \int_{\theta=0}^{\pi} \int_{\rho=0}^{1} \int_{z=0}^{1-\rho^2} (3\rho\cos\theta + 6\rho\sin\theta)\rho\, dz\, d\rho\, d\theta$$

$$= \int_0^{\pi} \int_0^1 \left(3\rho^2(1-\rho^2)\cos\theta + 6\rho^2(1-\rho^2)\sin\theta\right) d\rho\, d\theta$$

$$= 0 + 6\left[\frac{\rho^3}{3} - \frac{\rho^5}{5}\right]_0^1 \cdot \int_0^{\pi} \sin\theta\, d\theta = \cancel{6}^{2} \cdot \frac{2}{\cancel{15}_{5}} \cdot 2 = \frac{8}{5}$$

e

$$\iint_{S_2} \langle \vec{F}, \underbrace{d\vec{S}}_{(0,0,-1)\,dx\,dy} \rangle = -\iint_{S_2} (x\overset{\overset{0}{\|}}{z}+y)\,dx\,dy \overset{CP}{=} -\int_0^{\pi}\int_0^{1} (\rho\,\mathrm{sen}\,\theta)\,\rho\,d\rho\,d\theta$$

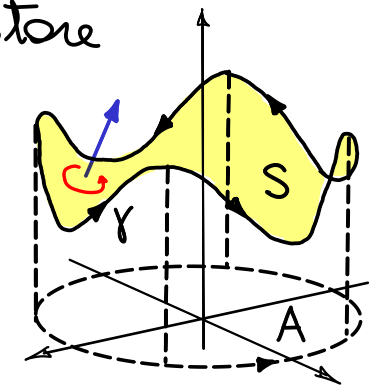
$$= -\left[\frac{\rho^3}{3}\right]_0^1 \int_0^{\pi} \mathrm{sen}\,\theta\,d\theta = -\frac{2}{3},$$

$$\iint_{S_3} \langle \vec{F}, \underbrace{d\vec{S}}_{(0,-1,0)\,dx\,dz} \rangle = -\iint_{S_3} (3\overset{\overset{0}{\|}}{y^2}+z)\,dx\,dz = -\int_{x=-1}^{1} \left(\int_{z=0}^{1-x^2} z\,dz \right) dx$$

$$= -\cancel{2}\int_0^1 \frac{(1-x^2)^2}{\cancel{2}}\,dx = -\left[x - 2\frac{x^3}{3} + \frac{x^5}{5}\right]_0^1$$

$$= -\left(1 - \frac{2}{3} + \frac{1}{5}\right) = -\frac{8}{15}.$$

$\boxed{3.a}$ Usando il teorema del rotore calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove



$\vec{F} = (\operatorname{sen} x, z(x-1), y(x+1))$ e

$\gamma = \{(x,y,z) : x^2+y^2=4,\ z = 2+x^2y^2\}$.

γ è orientata in modo che la sua proiezione su $z=0$ sia in verso antiorario.

Sia $S = \{(x,y,z) : x^2+y^2 \le 4,\ z = 2+x^2y^2\}$.

Parametrizzazione cartesiana di S:

$$\vec{\sigma}(x,y) = (x, y, 2+x^2y^2) \quad \text{con } A = \{(x,y) : x^2+y^2 \le 4\}.$$

Allora $\vec{\sigma}_x \times \vec{\sigma}_y = (-2xy^2, -2yx^2, 1)$ e $\vec{\sigma}$ induce su $\gamma = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\operatorname{rot}(\vec{F}) = \det\begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \operatorname{sen} x & z(x-1) & y(x+1) \end{bmatrix} = (2, -y, z).$$

Quindi

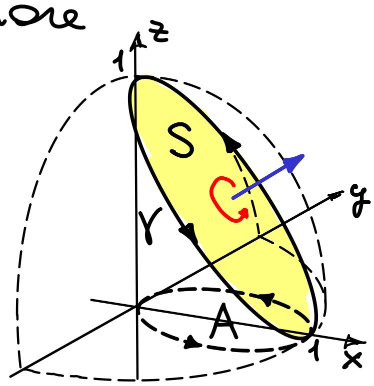
$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle \overset{TR}{=} \iint_S \langle \operatorname{rot}(\vec{F}), d\vec{S}\rangle$$

$$= \iint_A \langle (2, -y, \underset{}{\overset{2+x^2y^2}{\overset{\shortparallel}{z}}}), (-2xy^2, -2yx^2, 1)\rangle\, dx\,dy$$

$$= \iint_{\{x^2+y^2\le 4\}} (\overset{x\text{-dispari}}{-4xy^2} + 3x^2y^2 + 2)\, dx\,dy \overset{CP}{=} 3\int_0^{2\pi}\int_0^2 \rho^4 \overbrace{\cos^2\theta \cdot \operatorname{sen}^2\theta}^{\frac{1}{4}\operatorname{sen} 2\theta}\, \rho\, d\rho\, d\theta + 2\overset{4\pi^2}{\overset{\shortparallel}{|A|}}$$

$$= \frac{\cancel{3}}{4}\int_0^{2\pi} \operatorname{sen}^2 2\theta\, d\theta \left[\frac{\rho^6}{\cancel{6}_2}\right]_0^2 + 8\pi = \frac{1}{4}\pi \cdot 32 + 8\pi = 16\pi.$$

3.b Usando il teorema del rotore calcolare $\int_\gamma \langle \vec{F}, d\vec{s} \rangle$ dove



$\vec{F} = (y + ye^{xy}, xe^{xy}, x^3)$ e

$\gamma = \{(x,y,z) : x^2+y^2+z^2=1, x+z=1\}$

orientata in modo che la sua proiezione su $z=0$ sia in verso antiorario.

Sia $S = \{(x,y,z) : x^2+y^2+z^2 \le 1, x+z=1\}$.

Parametrizzazione cartesiana di S:

$\vec{\sigma}(x,y) = (x, y, 1-x)$ con $A = \{(x,y) : x^2+y^2+(1-x)^2 \le 1\}$.

Allora $\vec{\sigma_x} \times \vec{\sigma_y} = (1,0,1)$ e $\vec{\sigma}$ induce su $\gamma = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\mathrm{rot}(\vec{F}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y+ye^{xy} & xe^{xy} & x^3 \end{bmatrix} = (0, -3x^2, -1).$$

Quindi

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle \overset{TR}{=} \iint_S \langle \mathrm{rot}(\vec{F}), d\vec{S} \rangle$$

$$= \iint_A \langle (0,-3x^2,-1), (1,0,1) \rangle \, dx\, dy$$

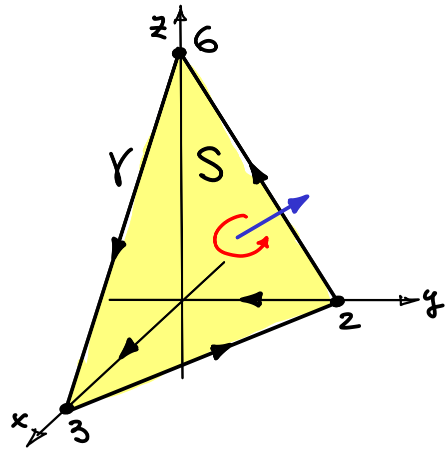
$$= -\iint_A dx\, dy = -|A| = -\pi \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{2}} = -\frac{\pi}{2\sqrt{2}}.$$

∂A è un'ellisse di semiassi $\frac{1}{2}$ e $\frac{1}{\sqrt{2}}$.

$x^2 + y^2 + (1-x)^2 = 1 \iff 2x^2 - 2x + y^2 = 0$

$\iff 2\left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{2} \iff \frac{\left(x-\frac{1}{2}\right)^2}{(1/2)^2} + \frac{y^2}{(1/\sqrt{2})^2} = 1$

$\boxed{3.c}$ Usando il teorema del rotore calcolare $\int_\gamma \langle \vec{F}, d\vec{s}\rangle$ dove $\vec{F}=(z+1, xz, y^2-yz)$ e γ è il bordo del triangolo di vertici $(3,0,0),(0,2,0),(0,0,6)$ orientato in modo che la sua proiezione sul piano $z=0$ sia in verso antiorario.



Il piano che passa per i tre punti dati:

$$\frac{x}{3}+\frac{y}{2}+\frac{z}{6}=1 \quad \text{ossia} \quad 2x+3y+z=6.$$

Parametrizzazione cartesiana di S tale che $\partial^+ S=\gamma$:

$\vec{\sigma}(x,y)=(x,y,6-2x-3y)$ con $A=\{(x,y): x,y\geq 0,\ 2x+3y\leq 6\}$.

Allora $\vec{\sigma_x}\times\vec{\sigma_y}=(2,3,1)$ che induce su $\gamma=\partial^+S$ l'orientazione richiesta. Calcolo del rotore:

$$\mathrm{rot}(\vec{F})=\det\begin{bmatrix}\vec{i} & \vec{j} & \vec{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ z+1 & xz & y^2-yz\end{bmatrix}=(2y-z-x,1,z).$$

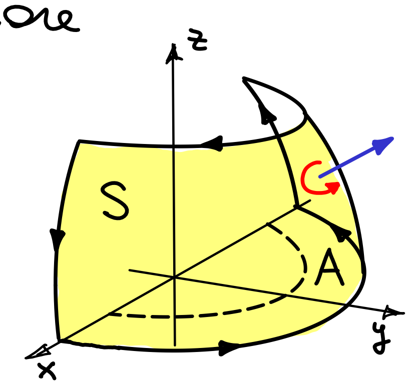
Quindi

$$\int_\gamma \langle \vec{F}, d\vec{s}\rangle \overset{TR}{=} \iint_S \langle \mathrm{rot}(\vec{F}), d\vec{S}\rangle$$

$$=\iint_A \langle (2y-\overset{6-2x-3y}{z}-x, 1, \overset{6-2x-3y}{z}), (2,3,1)\rangle dx\,dy$$

$$=\int_{x=0}^{3}\left(\int_{y=0}^{2-\frac{2x}{3}}(-3+7y)\,dy\right)dx=(-3+7\underset{\frac{2}{3}=\frac{1}{3}(0+2+0)}{\bar{y}})\underset{}{\overset{\frac{2\cdot 3}{2}=3}{|A|}}=-9+14=5.$$

$\boxed{4.a}$ Usando il teorema del rotore calcolare $\int_{\partial^+ S} \langle \vec{F}, d\vec{s} \rangle$ dove



$\vec{F} = (e^x, xy+2x, x^2+z^2)$ e

$S = \{(x,y,z) : x^2+y^2 = 4-z, 0 \leq z \leq 3, y \geq 0\}$

orientata in modo che $\langle \vec{n}, \vec{k} \rangle \geq 0$.

Parametrizzazione cartesiana di S:

$\vec{\sigma}(x,y) = (x, y, 4-x^2-y^2)$ con $A = \{(x,y) : 1 \leq x^2+y^2 \leq 4, y \geq 0\}$.

Allora $\vec{\sigma}_x \times \vec{\sigma}_y = (2x, 2y, \underset{}{\overset{\geq 0}{1}})$ e $\vec{\sigma}$ induce su $\gamma = \partial^+ S$ l'orientazione richiesta. Calcolo del rotore:

$$\mathrm{rot}(\vec{F}) = \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & xy+2x & x^2+z^2 \end{bmatrix} = (0, -2x, y+2).$$

Quindi

$$\int_\gamma \langle \vec{F}, d\vec{s} \rangle \overset{TR}{=} \iint_S \langle \mathrm{rot}(\vec{F}), d\vec{S} \rangle$$

$$= \iint_A \langle (0,-2x,y+2), (2x,2y,1) \rangle \, dx\, dy$$

x-dispari

$$= \iint_A (-4xy + y + 2) \, dx\, dy$$

simmetrico per $x=0$

$$\overset{CP}{=} \int_{\theta=0}^{\pi} \int_{\rho=1}^{2} \rho \sin\theta \, \rho \, d\rho \, d\theta + 2\underbrace{|A|}_{=(4\pi-\pi)/2}$$

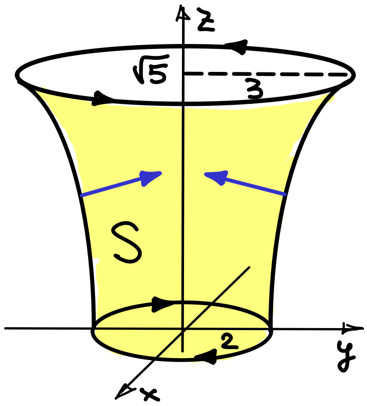
$$= \Big[-\cos\theta\Big]_0^\pi \cdot \left[\frac{\rho^3}{3}\right]_1^2 + 3\pi = 2\,\frac{8-1}{3} + 3\pi = \frac{14}{3} + 3\pi.$$

4.b Usando il teorema del rotore calcolare $\int_{\partial^+ S} \langle \vec{F}, d\vec{s} \rangle$ dove

$$\vec{F} = \left(-\frac{y}{x^2+y^2}, \frac{x}{x^2+y^2} + xy^2, z\right) \text{ e}$$

$$S = \left\{(x,y,z) : x^2+y^2 = z^2+4,\ 0 \le z \le \sqrt{5}\right\}$$

orientata in modo che $\langle \vec{n}, \vec{k} \rangle \geq 0$.



Notiamo che S non interseca l'asse z dove $\vec{F}$ non è definito.

Parametrizzazione cartesiana di S:

$$\vec{\sigma}(x,y) = \left(x, y, \sqrt{x^2+y^2-4}\right) \text{ con } A = \left\{(x,y) : 4 \le x^2+y^2 \le 9\right\}$$

Allora

$$\vec{\sigma_x} \times \vec{\sigma_y} = \left(-\frac{x}{\sqrt{x^2+y^2-4}}, -\frac{y}{\sqrt{x^2+y^2-4}}, \underbrace{1}_{>0}\right)$$

che induce su S l'orientazione richiesta.

Calcolo del rotore:

$$\operatorname{rot}(\vec{F}) = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} + xy^2 & z \end{bmatrix} = (0, 0, y^2).$$

Quindi

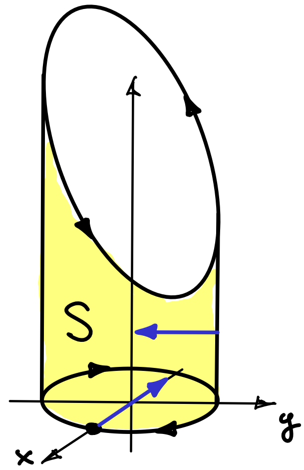
$$\int_{\partial^+ S} \langle \vec{F}, d\vec{s} \rangle \overset{TR}{=} \iint_S \langle \operatorname{rot}(\vec{F}), d\vec{S} \rangle$$

$$= \iint_A \langle (0,0,y^2), (\cdots, \cdots, 1) \rangle \, dx\,dy$$

$$= \iint_A y^2 \, dx\,dy \overset{CP}{=} \int_{\theta=0}^{2\pi} \int_{\rho=2}^{3} \rho^2 \sin^2\theta \, \rho \, d\rho \, d\theta$$

$$= \pi \left[\frac{\rho^4}{4}\right]_2^3 = \frac{\pi}{4}(81-16) = \frac{65\pi}{4}.$$

$\boxed{4.C}$ Usando il teorema del rotore calcolare $\int_{\partial^+ S} \langle \vec{F}, d\vec{s} \rangle$ dove



$\vec{F} = (z^2 + x^2 y, 3xyz, 2xz)$ e

$S = \{(x,y,z) : x^2 + y^2 = 1, 0 \leq z \leq 3 - x - y\}$

orientata in modo che $\vec{n} = (-1,0,0)$ in $(1,0,0)$.

Parametrizzazione di S:

$$\vec{\sigma}(\theta, z) = (\cos\theta, \operatorname{sen}\theta, z)$$

con $A = \{(\theta, z) : \theta \in [0, 2\pi], 0 \leq z \leq 3 - \cos\theta - \operatorname{sen}\theta\}$.

Così $\vec{\sigma_\theta} \times \vec{\sigma_z} = (\cos\theta, \operatorname{sen}\theta, 0)$ induce su S l'orientazione opposta perché per $\theta = 0$ e $z = 0$, ossia nel punto $(1,0,0)$, $\vec{n}^+ = (1,0,0)$.

Calcolo del rotore:

$$\operatorname{rot}(\vec{F}) = \det \begin{bmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z^2 + x^2 y & 3xyz & 2xz \end{bmatrix} = (-3xy, 0, 3yz - x^2).$$

Quindi

$$\int_{\partial^+ S} \langle \vec{F}, d\vec{s} \rangle \overset{TR}{=} \iint_S \langle \operatorname{rot}(\vec{F}), d\vec{S} \rangle$$

$$= -\iint_A \langle (-3xy, 0, 3yz - x^2), (\cos\theta, \operatorname{sen}\theta, 0) \rangle \, d\theta \, dz$$

orientazione

$$= \int_{\theta=0}^{2\pi} \left(\int_{z=0}^{3-\cos\theta-\operatorname{sen}\theta} 3\cos^2\theta \operatorname{sen}\theta \, dz \right) d\theta$$

θ-dispari

$$= \int_0^{2\pi} \left(9\cos^2\theta \operatorname{sen}\theta - 3\cos^3\theta \operatorname{sen}\theta - 3\overbrace{\cos^2\theta \operatorname{sen}^2\theta}^{\operatorname{sen}^2(2\theta)/4} \right) d\theta$$

$\int_{-\pi}^{\pi}$

$$= 0 + 0 - \frac{3}{4} \int_0^{2\pi} \operatorname{sen}^2(2\theta) \, d\theta = -\frac{3\pi}{4}.$$