

ANALISI MATEMATICA 2 - FOGLIO 5

1.a

$$f(x, y, z) = x + z \quad \text{Max/min in}$$

$$D = \left\{ (x, y, z) : \begin{array}{l} x^2 + y^2 + z^2 = 2 \\ z = x^2 + y^2 \end{array} \right\}$$

$f \in C^2(\mathbb{R}^3)$, D è chiuso e limitato. Possiamo

$$g_1(x, y, z) = x^2 + y^2 + z^2 - 2 = 0, \quad g_2(x, y, z) = x^2 + y^2 - z = 0.$$

Regolarità:

$$\begin{bmatrix} g_{1x} & g_{1y} & g_{1z} \\ g_{2x} & g_{2y} & g_{2z} \end{bmatrix} = \begin{bmatrix} 2x & 2y & 2z \\ 2x & 2y & -1 \end{bmatrix}$$

Il rango non vale 2 se e solo se

$$\begin{cases} \det \begin{bmatrix} 2x & 2z \\ 2x & -1 \end{bmatrix} = 2x(-1-2z) = 0 \\ \det \begin{bmatrix} 2y & 2z \\ 2y & -1 \end{bmatrix} = 2y(-1-2z) = 0 \end{cases} \iff \begin{array}{l} (0, 0, z) \notin D \\ (x, y, -\frac{1}{2}) \notin D \end{array}$$

Tutti i punti di D sono regolari.

Moltiplicatori:

$$\begin{cases} 1 = \lambda 2x + \mu 2x \rightarrow 1 = 2x(\lambda + \mu) \rightarrow \lambda + \mu \neq 0 \\ 0 = \lambda 2y + \mu 2y \rightarrow 2y(\lambda + \mu) = 0 \rightarrow y = 0 \\ 1 = \lambda 2z - \mu \\ x^2 + y^2 + z^2 = 2 \\ z = x^2 + y^2 \end{cases}$$

Punti stazionari vincolati:

$$(1, 0, 1) \quad \lambda = \frac{1}{2}, \mu = 0$$

$$(-1, 0, 1) \quad \lambda = \frac{1}{6}, \mu = -\frac{2}{3}$$

$$\boxed{\begin{array}{l} f(1, 0, 1) = 2 \quad \text{Max} \\ f(-1, 0, 1) = 0 \quad \text{Min} \end{array}}$$

$(1, 0, 1)$ punto di max assoluto

$(-1, 0, 1)$ punto di min assoluto

$$\begin{cases} y = 0 \\ 1 = 2(\lambda + \mu)x \\ 1 = 2\lambda z - \mu \\ x^2 + z^2 = 2 \rightarrow z + z^2 = 2 \\ z = x^2 \end{cases}$$

$z = 1 \rightarrow x = 1, x = -1$
 $z = -2 = x^2 \rightarrow \emptyset$

1.6 $f(x,y,z) = 2x + 2y - z$ Max/min in
 $D = \{(x,y,z) : x^2 + y^2 + z^2 \leq 4\}$

$f \in C^2(\mathbb{R}^3)$, D è chiuso e limitato.

Dato che $\nabla f(x,y,z) = (2, 2, -1) \neq (0, 0, 0)$

non ci sono punti stazionari interni a D .

Bordo: $\partial D = \{(x,y,z) : x^2 + y^2 + z^2 = 4\}$

Sia $g(x,y,z) = x^2 + y^2 + z^2 - 4 = 0$. Allora

$$\nabla g(x,y,z) = (2x, 2y, 2z) = (0, 0, 0)$$

se e solo se $(x,y,z) = (0, 0, 0) \notin \partial D$.

I punti di D sono tutti regolari.

Moltiplicatori:

$$\begin{cases} 2 = \lambda 2x & \xrightarrow{\lambda \neq 0} x = \frac{1}{\lambda} \\ 2 = \lambda 2y & \rightarrow y = \frac{1}{\lambda} \\ -1 = \lambda 2z & \rightarrow z = -\frac{1}{2\lambda} \\ x^2 + y^2 + z^2 = 4 & \rightarrow \frac{1}{\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 4 \rightarrow \lambda^2 = \frac{9}{16} \end{cases} \begin{matrix} \nearrow \lambda = \frac{3}{4} \\ \searrow \lambda = -\frac{3}{4} \end{matrix}$$

Punti stazionari vincolati:

$$\left(\frac{4}{3}, \frac{4}{3}, -\frac{2}{3}\right) \quad \lambda = \frac{3}{4} \quad f\left(\frac{4}{3}, \frac{4}{3}, -\frac{2}{3}\right) = 6 \quad \text{Max}$$

$$\left(-\frac{4}{3}, -\frac{4}{3}, \frac{2}{3}\right) \quad \lambda = -\frac{3}{4} \quad f\left(-\frac{4}{3}, -\frac{4}{3}, \frac{2}{3}\right) = -6 \quad \text{Min}$$

$\left(\frac{4}{3}, \frac{4}{3}, -\frac{2}{3}\right)$ punto di max. assoluto

$\left(-\frac{4}{3}, -\frac{4}{3}, \frac{2}{3}\right)$ punto di min. assoluto

1.c $f(x,y,z) = 4x + z^2$ Max/min in $D = \{(x,y,z) : x^2 + y^2 \leq z \leq 4\}$

$f \in C^2(\mathbb{R}^3)$, D è chiuso e limitato.

Dato che $\nabla f(x,y,z) = (4, 0, 2z) \neq (0,0,0)$ non ci sono punti stazionari interni a D .

Bordo: $\partial D = \Gamma_1 \cup \Gamma_2$ dove

$$\Gamma_1 = \{(x,y,z) : z=4, x^2+y^2 \leq 4\} \text{ e } \Gamma_2 = \{(x,y,z) : x^2+y^2 = z, z < 4\}$$

1) Restringendo f a Γ_1 abbiamo che $z=4$ e

$$h(x,y) = f(x,y,4) = 4x + 16.$$

Studiando $4x+16$ nel cerchio $x^2+y^2 \leq 4$ si ha che

$$8 = h(-2,0) \leq h(x,y) = 4x+16 \leq h(2,0) = 24.$$

Dunque i punti rilevanti in Γ_1 sono

$$(-2,0,4) \text{ e } (2,0,4) \quad f(-2,0,4) = 8, f(2,0,4) = 24$$

2) Su Γ_2 usiamo i moltiplicatori con

$$g(x,y,z) = x^2 + y^2 - z = 0.$$

Allora $\nabla g(x,y,z) = (2x, 2y, -1) \neq (0,0,0)$ e tutti i

punti di Γ_2 sono regolari

$$\begin{cases} 4 = \lambda 2x & \rightarrow \lambda \neq 0 \\ 0 = \lambda 2y & \rightarrow y = 0 \\ 2z = \lambda(-1) \\ x^2 + y^2 = z \end{cases} \begin{cases} y = 0 \\ x = 2/\lambda \\ z = -\lambda/2 \\ \frac{4}{\lambda^2} + 0^2 = -\frac{\lambda}{2} \rightarrow \lambda^3 = -8 \rightarrow \lambda = -2 \end{cases}$$

ok perché $z < 4$
 $x = -1, z = 1$
 $\uparrow$

L'unico punto stazionario è $(-1,0,1) \in \Gamma_2$ e

$$f(-1,0,1) = -3. \text{ Quindi}$$

$(-1,0,1)$ punto di min. assoluto

$(2,0,4)$ punto di max. assoluto

2.a $\iint_D \frac{e^{x/y}}{y^4} dx dy \quad D=[0,2] \times [1,2]$

$$= \int_{y=1}^2 \frac{1}{y^4} \left(\int_{x=0}^2 e^{x/y} dx \right) dy = \int_{y=1}^2 \frac{1}{y^4} \left[y e^{x/y} \right]_0^2 dy = \int_1^2 \frac{e^{2/y} - 1}{y^3} dy$$

$t = \frac{1}{y} \rightarrow dt = -\frac{dy}{y^2}$

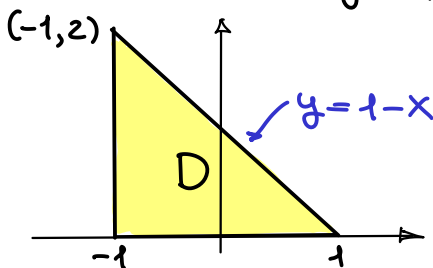
$$= \int_1^{1/2} \frac{e^{2t} - 1}{1/t} (-dt) = \int_{1/2}^1 (t e^{2t} - t) dt$$

$$= \left[\frac{t e^{2t}}{2} - \frac{e^{2t}}{4} - \frac{t^2}{2} \right]_{1/2}^1 = \frac{e^2}{2} - \frac{e^2}{4} - \frac{1}{2} - \left(\frac{e}{4} - \frac{e}{4} - \frac{1}{8} \right) = \frac{e^2}{4} - \frac{3}{8}$$

dove $\int t e^{2t} dt = \int t d\left(\frac{e^{2t}}{2}\right) = \frac{t e^{2t}}{2} - \frac{e^{2t}}{4} + c.$

2.b $\iint_D (1 - |x| - y) dx dy \quad D = \{(x,y) : x \geq -1, y \geq 0, x+y \leq 1\}$

$$= \int_{x=-1}^1 \left(\int_{y=0}^{1-x} (1 - |x| - y) dy \right) dx$$



$$= \int_{-1}^1 \left[(1 - |x|)y - \frac{y^2}{2} \right]_0^{1-x} dx = \int_{-1}^1 \left((1 - |x|)(1-x) - \frac{1}{2}(1-x)^2 \right) dx$$

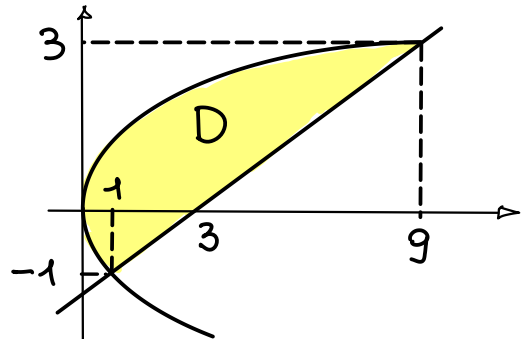
$\hookrightarrow \frac{1}{P} - \frac{|x|}{P} - \frac{x}{P} + \frac{x|x|}{P} - \frac{1}{2P} + \frac{x}{P} - \frac{x^2}{2P}$

$$= 2 \int_0^1 \left(\frac{1}{2} - x - \frac{x^2}{2} \right) dx = \left[x - x^2 - \frac{x^3}{3} \right]_0^1 = -\frac{1}{3}.$$

2.c

$$\iint_D \frac{1}{(2y-x+8)^2} dx dy \quad D = \{(x,y): y^2 \leq x \leq 2y+3\}$$

$$= \int_{y=-1}^3 \left(\int_{x=y^2}^{2y+3} \frac{1}{(2y-x+8)^2} dx \right) dy$$



$$= \int_{y=-1}^3 \left[\frac{1}{2y-x+8} \right]_{y^2}^{2y+3} dy = \int_{-1}^3 \left(\frac{1}{5} - \frac{1}{2y-y^2+8} \right) dy$$

$-(y+2)(y-4)$

$$= \frac{3-(-1)}{5} + \frac{1}{6} \int_{-1}^3 \left(\frac{1}{y-4} - \frac{1}{y+2} \right) dy = \frac{4}{5} + \frac{1}{6} \left[\log \left| \frac{y-4}{y+2} \right| \right]_{-1}^3$$

$$= \frac{4}{5} + \frac{1}{6} (\log(\frac{1}{5}) - \log(5)) = \frac{4}{5} - \frac{\log(5)}{3}$$

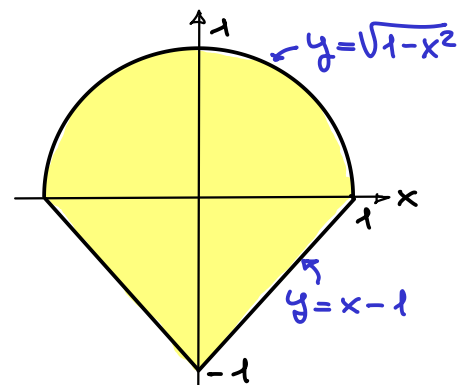
2.d

$$\iint_D \frac{x+y}{1+x^2} dx dy \quad D = \{(x,y): y \geq |x|-1, x^2+y^2 \leq 1\}$$

D è simmetrico rispetto a $x=0$.

$$= \iint_D \left(\overset{x\text{-disp}}{\frac{x}{1+x^2}} + \overset{x\text{-pau}}{\frac{y}{1+x^2}} \right) dx dy$$

$$= 0 + 2 \iint_{D \cap \{x \geq 0\}} \frac{y}{1+x^2} dx dy = 2 \int_{x=0}^1 \frac{1}{1+x^2} \left(\int_{x-1}^{\sqrt{1-x^2}} y dy \right)$$



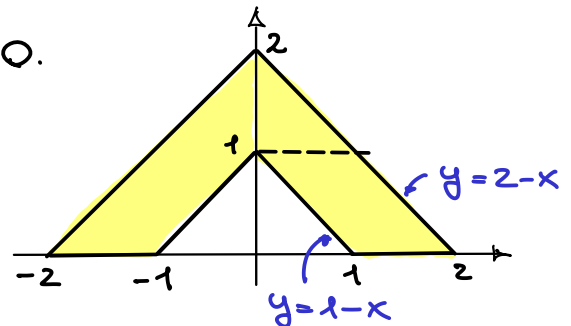
$$\begin{aligned}
&= \cancel{2} \int_0^1 \frac{1}{1+x^2} \left[\frac{\cancel{y^2}}{\cancel{2}} \right]_{x-1}^{\sqrt{1-x^2}} = \int_0^1 \frac{\cancel{1-x^2} - x^2 + 2x - \cancel{1}}{1+x^2} dx \\
&= \int_0^1 \left(\frac{2x}{1+x^2} - 2 \frac{x^2 \pm 1}{1+x^2} \right) dx = \left[\log(1+x^2) - 2x + 2 \operatorname{arctg}(x) \right]_0^1 \\
&= \log 2 - 2 + \frac{\pi}{2}.
\end{aligned}$$

2.e $\iint_D y(1-\sin(x)) dx dy \quad D = \{(x,y): y \geq 0, 1 \leq |x|+y \leq 2\}$

D è simmetrico rispetto a $x=0$.

$$= \iint_D y dx dy - \iint_D y \sin x dx dy$$

x -~~sen~~ x -~~dispen~~



$$= 2 \iint_{D \cap \{x \geq 0\}} y dx dy + 0$$

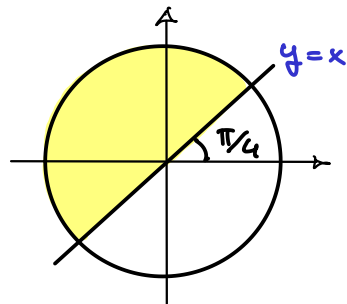
$$= 2 \int_{y=0}^1 y \int_{x=1-y}^{2-y} dx + 2 \int_{y=1}^2 y \int_{x=0}^{2-y} dx$$

$$= 2 \int_0^1 y(2-\cancel{y}-1+\cancel{y}) dy + 2 \int_1^2 y(2-y) dy$$

$$= \left[y^2 \right]_0^1 + 2 \left[y^2 - \frac{y^3}{3} \right]_1^2 = 1 + \frac{4}{3} = \frac{7}{3}.$$

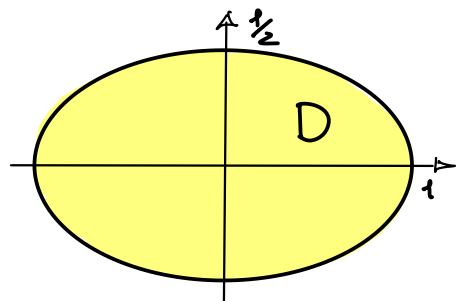
2. $\iint_D \sqrt{1-x^2-y^2} dx dy$ $D = \{ (x,y) : y \geq x, x^2+y^2 \leq 1 \}$

$$\begin{aligned}
 C_P &= \int_{\rho=0}^1 \sqrt{1-\rho^2} \left(\int_{\theta=\frac{\pi}{4}}^{\pi+\frac{\pi}{4}} d\theta \right) \rho d\rho \\
 &= \pi \int_0^1 \sqrt{1-\rho^2} \rho d\rho = \pi \int_0^1 \sqrt{t} \cdot \frac{dt}{2} \\
 &= \frac{\pi}{2} \left[\frac{2}{3} t^{3/2} \right]_0^1 = \frac{\pi}{3}.
 \end{aligned}$$



2.9 $\iint_D (x-1)^2 dx dy$ $D = \{(x,y) : x^2 + 4y^2 \leq 1\}$
 $\downarrow$ $x^2 - 2x + 1$ $\uparrow$ simmetrico
 $\uparrow$ $\text{rispetto a } x=0$

$$= \iint_D (x^2 + 1) dx dy$$



Con il cambio di coordinate

$$\overline{\Phi}: \begin{cases} x = r \cos \theta \\ y = \frac{1}{2} r \sin \theta \end{cases} \quad \text{so the the } \overline{\Phi}^{-1}(D) = [0, 1] \times [0, 2\pi]$$

$$e \quad |\det J_{\bar{\phi}}(p, \theta)| = \left| \det \begin{bmatrix} \cos \theta & -p \sin \theta \\ \frac{1}{2} \sin \theta & \frac{1}{2} p \cos \theta \end{bmatrix} \right| = \frac{1}{2} p.$$

Così l'integrale diventa

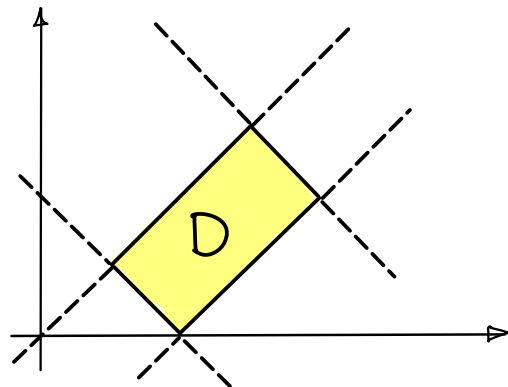
$$= \int_0^{2\pi} \int_0^1 (p^2 \cos^2 \theta + 1) \frac{1}{2} p \, dp \, d\theta = \frac{1}{2} \int_0^{2\pi} \left[\frac{p^4}{4} \cos^2 \theta + \frac{p^2}{2} \right]_0^1 d\theta$$
$$= \frac{1}{8} \int_0^{2\pi} \cos^2 \theta \, d\theta + \frac{2\pi}{4} = \frac{5\pi}{8} . \quad \int_0^{2\pi} \cos^2 \theta \, d\theta = \pi$$

2.8

$$\iint_D (x-y) \log(x+y) dx dy \quad D = \{(x,y): 0 \leq x-y \leq 1, 1 \leq x+y \leq 3\}$$

Con il cambio di coordinate

$$\overline{\Phi}^{-1}: \begin{cases} u = x-y \\ v = x+y \end{cases} \Rightarrow \overline{\Phi}: \begin{cases} x = \frac{u+v}{2} \\ y = \frac{-u+v}{2} \end{cases}$$



in che $\overline{\Phi}^{-1}(D) = [0, 1] \times [1, 3]$ e

$$|\det J_{\overline{\Phi}}(u,v)| = \left| \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix} \right| = \frac{1}{2}.$$

Così l'integrale diventa

$$= \int_{u=0}^1 \int_{v=1}^3 u \log(v) \frac{1}{2} du dv = \frac{1}{2} \left[\frac{u^2}{2} \right]_0^1 \left[v \log(v) - v \right]_1^3$$

$$= \frac{1}{4} (3 \log 3 - 3 - (0 - 1)) = \frac{1}{4} (3 \log 3 - 2).$$

3.a $D = \left\{ (x, y, z) : \frac{|x|}{4} + \frac{|y|}{3} + |z| \leq 1 \right\}$ Volume?

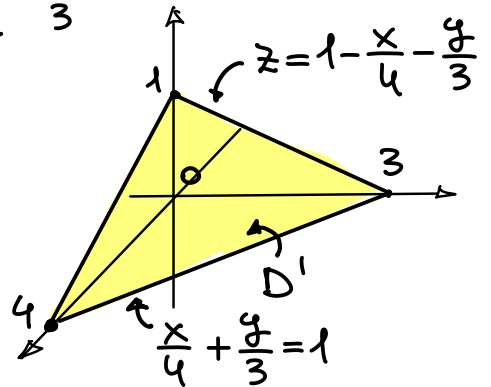
D è simmetrico rispetto ai piani $x=0, y=0, z=0$
 pertanto $|D| = 8|E|$ dove

$$E = \left\{ (x, y, z) : \frac{x}{4} + \frac{y}{3} + z \leq 1, x, y, z \geq 0 \right\}$$

1) Per fili con $\varphi_1(x, y) = 0, \varphi_2(x, y) = 1 - \frac{x}{4} - \frac{y}{3}$

e $D' = \left\{ (x, y) : \frac{x}{4} + \frac{y}{3} \leq 1, x, y \geq 0 \right\}$.

$$|D| = 8|E| = 8 \iint_{D'} \left(1 - \frac{x}{4} - \frac{y}{3} \right) dx dy$$



$$= 8 \int_{x=0}^4 \left(\int_{y=0}^{3-\frac{3x}{4}} \left(1 - \frac{x}{4} - \frac{y}{3} \right) dy \right) dx = 8 \int_{u=0}^1 \left(\int_{v=0}^{1-u} (1-u-v) 3 dv \right) 4 du$$

$\begin{cases} x=4u \\ y=3v \end{cases}$

$$= 8 \cdot 12 \int_0^1 \left[v - uv - \frac{v^2}{2} \right]_0^{1-u} du = 8 \cdot 12 \int_0^1 \left((1-u)^2 - \frac{(1-u)^2}{2} \right) du$$

$$= \frac{8 \cdot 12}{2} \int_0^1 (1-u)^2 du = 48 \left[-\frac{(1-u)^3}{3} \right]_0^1 = \frac{48}{3} = 16.$$

2) Per sezioni con $z \in [0, 1]$ e $S_z = \left\{ (x, y) : \frac{x}{4} + \frac{y}{3} \leq 1-z \right\}$

$$|D| = 8|E| = 8 \int_{z=0}^1 \left(\iint_{S_z} 1 dx dy \right) dz$$

triangolo rettangolo con
 cateti $4(1-z)$ e $3(1-z)$

$$= 8 \int_0^1 |S_z| dz = 8 \int_0^1 \frac{4 \cdot 3 (1-z)^2}{2} dz = 8 \cdot 6 \left[-\frac{(1-z)^3}{3} \right]_0^1$$

$$= 8 \cdot \frac{1}{3} = 16.$$

3.6

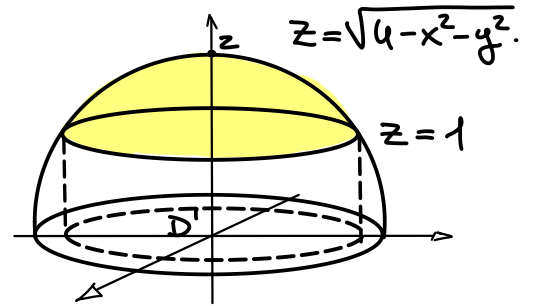
$$D = \{(x, y, z) : x^2 + y^2 + z^2 \leq 4, z \geq 1\} \quad \text{Volume?}$$

calotta

1) Per fili con $D' = \{x^2 + y^2 + 1 \leq 4\}$

$$\varphi_1(x, y) = 1 \quad \text{e} \quad \varphi_2(x, y) = \sqrt{4 - x^2 - y^2}.$$

$$|D| = \iint_{D'} (\sqrt{4 - x^2 - y^2} - 1) \, dx \, dy$$



$$\stackrel{\text{CP}}{=} \int_{\rho=0}^{\sqrt{3}} \int_{\theta=0}^{2\pi} (\sqrt{4 - \rho^2} - 1) \rho \, d\rho \, d\theta = 2\pi \int_0^{\sqrt{3}} (\sqrt{4 - \rho^2} - 1) \rho \, d\rho$$

$t = 4 - \rho^2 \rightarrow dt = -2\rho d\rho$

$$= 2\pi \int_4^1 (t^{1/2} - 1) \left(-\frac{dt}{2}\right) = \pi \left[\frac{2}{3} t^{3/2} - t \right]_1^4$$

$$= \pi \left(\left(\frac{16}{3} - 4 \right) - \left(\frac{2}{3} - 1 \right) \right) = \frac{5}{3} \pi.$$

2) Per sezioni con $z \in [1, 2]$ e $S_z = \{(x, y) : x^2 + y^2 \leq 4 - z^2\}$
cerchio di raggio $\sqrt{4 - z^2}$

$$|D| = \int_{z=1}^2 \left(\iint_{S_z} 1 \, dx \, dy \right) dz = \int_1^2 |S_z| \, dz =$$

$$= \int_1^2 \pi(4 - z^2) \, dz = \pi \left[4z - \frac{z^3}{3} \right]_1^2 = \pi \left(8 - \frac{8}{3} - 4 + \frac{1}{3} \right)$$

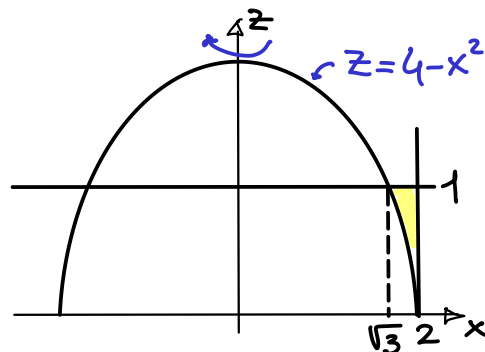
$$= \frac{5\pi}{3}$$

3.c $D = \{(x, y, z) : 0 \leq 4 - x^2 - y^2 \leq z \leq 1\}$ Volume?

È ottenuto ruotando intorno all'asse z la regione piana

$$\{(x, z), \sqrt{3} \leq x \leq 2, 4 - x^2 \leq z \leq 1\}$$

$$\begin{cases} z = 4 - x^2 \\ z = 1 \\ x \geq 0 \end{cases} \Rightarrow x = \sqrt{3}$$



$$\begin{aligned} |D| &\stackrel{CC}{=} \int_0^{2\pi} d\theta \cdot \int_{p=\sqrt{3}}^2 p \left(\int_{z=4-p^2}^1 dz \right) dp = 2\pi \int_{\sqrt{3}}^2 p [z]_{4-p^2}^1 dp \\ &= 2\pi \int_{\sqrt{3}}^2 p (1 - (4 - p^2)) dp = 2\pi \int_0^1 (1 - t) \frac{dt}{2} = \pi \left[t - \frac{t^2}{2} \right]_0^1 = \frac{\pi}{2}. \end{aligned}$$

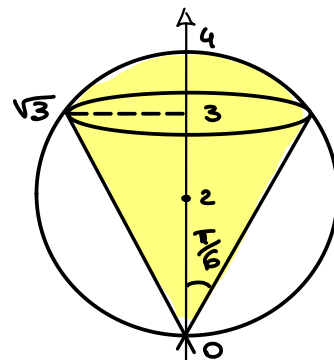
3.d $D = \{(x, y, z) : \begin{matrix} 3(x^2 + y^2) \leq z^2 \\ x^2 + y^2 + (z - 2)^2 \leq 4 \end{matrix}\}$ Volume?

Coordinate sferiche:

$$\vec{\Phi}^{-1}(D) = \left\{ (p, \theta, \varphi) : \begin{matrix} \varphi \in [0, \pi/6] \\ \theta \in [0, 2\pi) \\ p \in [0, 4\cos\varphi] \end{matrix} \right\}$$

$$x^2 + y^2 + z^2 - 4z + 4 \leq 4,$$

$$p^2 - 4p\cos\varphi \leq 0, \quad p \leq 4\cos\varphi$$



$$|D| = \iiint_{\vec{\Phi}^{-1}(D)} p^2 \sin\varphi \, dp \, d\theta \, d\varphi = 2\pi \int_0^{\pi/6} \sin\varphi \left(\int_0^{4\cos\varphi} p^2 dp \right) d\varphi$$

$$= 2\pi \int_0^{\pi/6} \sin\varphi \left[\frac{p^3}{3} \right]_0^{4\cos\varphi} d\varphi = \frac{2 \cdot 4^3 \pi}{3} \int_0^{\pi/6} (-\cos^3\varphi) d(\cos\varphi)$$

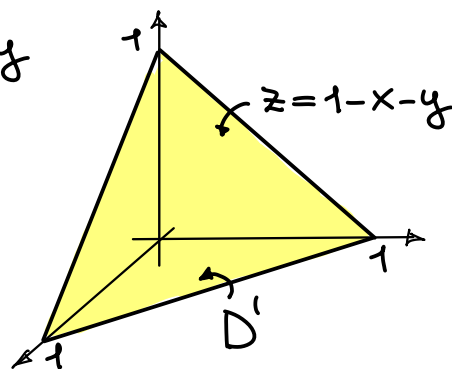
$$= \frac{2 \cdot 4^3 \pi}{3} \left[-\frac{\cos^4\varphi}{4} \right]_0^{\pi/6} = \frac{32\pi}{3} \left(1 - \left(\frac{\sqrt{3}}{2} \right)^4 \right) = \frac{32\pi}{3} \cdot \frac{7}{16} = \frac{14\pi}{3}.$$

4.a

$$\iiint_D \frac{1}{(x+y+z+1)^3} dx dy dz$$

dove $D = \{(x, y, z) : x, y, z \geq 0, x+y+z \leq 1\}$.

Per fili con $\varphi_1(x, y) = 0$, $\varphi_2(x, y) = 1 - x - y$
e $D' = \{(x, y) : x, y \geq 0, x+y \leq 1\}$



$$= \iint_{D'} \left(\int_{z=0}^{1-x-y} \frac{1}{(x+y+z+1)^3} dz \right) dx dy$$

$$= \iint_{D'} \left[\frac{-1/2}{(x+y+z+1)^2} \right]_0^{1-x-y} dx dy = \frac{1}{2} \iint_{D'} \left(\frac{1}{(x+y+1)^2} - \frac{1}{4} \right) dx dy$$

$$= \frac{1}{2} \int_{x=0}^1 \left(\int_{y=0}^{1-x} \frac{1}{(x+y+1)^2} dy \right) dx - \frac{|D'|}{8} = \frac{1}{2}$$

$$= \frac{1}{2} \int_0^1 \left[-\frac{1}{x+y+1} \right]_0^{1-x} dx - \frac{1}{16} = \frac{1}{2} \int_0^1 \left(\frac{1}{x+1} - \frac{1}{2} \right) dx - \frac{1}{16}$$

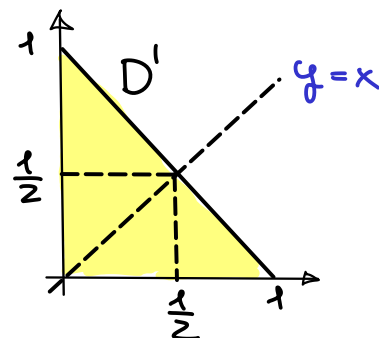
$$= \frac{1}{2} \left[\log(x+1) \right]_0^1 - \frac{1}{4} - \frac{1}{16} = \frac{\log(2)}{2} - \frac{5}{16}.$$

4.6 $\iiint_D |y-x| e^z dx dy dz$

dove $D = \{(x, y, z) : x, y, z \in [0, 1], z \leq x+y \leq 1\}$.

Per fare con $D' = \{(x, y) : x, y \geq 0 \text{ e } x+y \leq 1\}$ e
 $\varphi_1(x, y) = 0, \varphi_2(x, y) = x+y$.

$$= \iint_{D'} |y-x| \left(\int_{z=0}^{x+y} e^z dz \right) dx dy$$



$$= \iint_{D'} |y-x| (e^{x+y} - 1) dx dy$$

$$= \int_{x=0}^{1/2} \left(\int_{y=x}^{1-x} (y-x)(e^{x+y} - 1) dy \right) dx + \int_{y=0}^{1/2} \left(\int_{x=y}^{1-y} (x-y)(e^{x+y} - 1) dx \right) dy$$

UGUALI!

$$= 2 \int_{x=0}^{1/2} \left[(y-x-1) e^{x+y} - \frac{y^2}{2} + xy \right]_x^{1-x} dx$$

$$= 2 \int_{x=0}^{1/2} \left(-2xe - \frac{(1-x)^2}{2} + x(1-x) - \left(-e^{2x} - \frac{x^2}{2} + x^2 \right) \right) dx$$

$\frac{x^2}{2}$

$$= \left[-2ex^2 + \frac{(1-x)^3}{3} + x^2 - 2\frac{x^3}{3} + e^{2x} - \frac{x^3}{3} \right]_0^{1/2}$$

$$= -\frac{e}{2} + \frac{1}{24} + \frac{1}{4} - \frac{1}{8} + e - \frac{1}{3} - 1 = \frac{e}{2} - \frac{7}{6}.$$

$$\boxed{4.C} \quad \iiint_D z e^{x^2+y^2} dx dy dz$$

dove $D = \{(x, y, z) : x, y, z \geq 0, x^2 + y^2 + z^2 \leq 1\}$.

Per sezioni: per $z \in [0, 1]$

$$S_z = \{(x, y) : x^2 + y^2 \leq 1 - z^2, x \geq 0, y \geq 0\}.$$

così l'integrale diventa

$$= \int_{z=0}^1 z \left(\iint_{S_z} e^{x^2+y^2} dx dy \right) dz$$

$$\stackrel{CP}{=} \int_0^1 z \left(\int_{\rho=0}^{\sqrt{1-z^2}} \int_{\theta=0}^{\pi/2} e^{\rho^2} \rho d\rho d\theta \right) dz$$

$$= \frac{\pi}{2} \int_0^1 z \left[\frac{e^{\rho^2}}{2} \right]_0^{\sqrt{1-z^2}} dz = \frac{\pi}{4} \int_0^1 (e^{1-z^2} - 1) z dz$$

$$= \frac{\pi e}{4} \int_0^1 e^{-z^2} z dz - \frac{\pi}{4} \left[\frac{z^2}{2} \right]_0^1$$

$$= \frac{\pi e}{4} \left[-\frac{e^{-z^2}}{2} \right]_0^1 - \frac{\pi}{8}$$

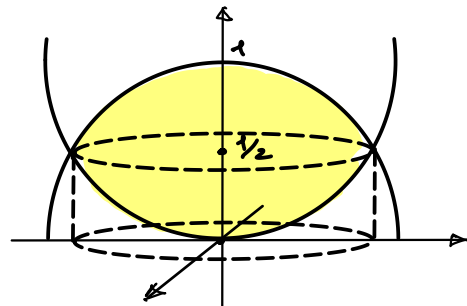
$$= \frac{\pi e}{8} (-e^{-1} + 1) - \frac{\pi}{8} = \frac{\pi}{8} (e - 2).$$

4.d $\iiint_D z^2 dx dy dz$

dove $D = \{(x, y, z): x^2 + y^2 \leq \min(1 - z^2, 2z - z^2)\}$

$$\begin{cases} x^2 + y^2 \leq 1 - z^2 \\ x^2 + y^2 \leq 2z - z^2 \end{cases} \iff \begin{cases} x^2 + y^2 + z^2 \leq 1 \\ x^2 + y^2 + (z - 1)^2 \leq 1 \end{cases}$$

D è l'intersezione di due palle di raggio 1 e centri $(0, 0, 0)$ e $(0, 0, 1)$



In coordinate cilindriche l'integrale si spezza sui due "gusci"

$$= \int_{z=0}^{1/2} z^2 \left(\int_{\theta=0}^{2\pi} \left(\int_{\rho=0}^{\sqrt{2z-z^2}} \rho d\rho \right) d\theta \right) dz + \int_{z=1/2}^1 z^2 \left(\int_{\theta=0}^{2\pi} \left(\int_{\rho=0}^{\sqrt{1-z^2}} \rho d\rho \right) d\theta \right) dz$$

$$= \int_0^{1/2} z^2 \cdot \cancel{2\pi} \cdot \frac{2z-z^2}{\cancel{2}} dz + \int_{1/2}^1 z^2 \cdot \cancel{2\pi} \cdot \frac{1-z^2}{\cancel{2}} dz$$

$$= \pi \left[\frac{\cancel{2} z^4}{\cancel{42}} - \frac{z^5}{5} \right]_0^{1/2} + \pi \left[\frac{z^3}{3} - \frac{z^5}{5} \right]_{1/2}^1$$

$$= \pi \left(\frac{1}{32} + \frac{1}{3} - \frac{1}{5} - \frac{1}{24} \right) = \frac{\pi 59}{480}.$$

4.e $\iiint_D |x-1| dx dy dz$

dove $D = \{(x, y, z): x^2 + y^2 + z^2 \leq 4, x \geq 0, z \geq 0\}$.

$$= \int_{x=0}^2 |x-1| \left(\iint_{\substack{y^2+z^2 \leq 4-x^2 \\ z \geq 0}} dy dz \right) dx = \int_0^2 |x-1| \cdot \left(\frac{\pi}{2} (4-x^2) \right) dx$$

Area

← semicerchio di raggio $\sqrt{4-x^2}$

$$= \frac{\pi}{2} \left(\int_0^1 (1-x)(4-x^2) dx - \int_1^2 (1-x)(4-x^2) dx \right)$$

$\rightarrow 4 - 4x - x^2 + x^3 \xrightarrow{\int} 4x - 2x^2 - \frac{x^3}{3} + \frac{x^4}{4}$

$$= \frac{\pi}{2} \left(2 \left(4 - 2 - \frac{1}{3} + \frac{1}{4} \right) - 0 - \left(8 - 8 - \frac{8}{3} + 4 \right) \right)$$

$$= \frac{15\pi}{12} = \frac{5\pi}{4}.$$