

ANALISI MATEMATICA - LEZIONE 17

ESEMPI

$$\bullet \lim_{x \rightarrow +\infty} x \left(\left(1 + \frac{1}{x} \right)^x - e \right) = ?$$

$$\text{Per } x \rightarrow +\infty, t = \frac{1}{x} \rightarrow 0^+ \text{ e}$$

$$x \left(\left(1 + \frac{1}{x} \right)^x - e \right) = \frac{(1+t)^{1/t} - e}{t} = \frac{1}{t} \left(\exp\left(\frac{\log(1+t)}{t}\right) - e \right)$$

$$\log(1+t) = t - \frac{t^2}{2} + o(t^2) \rightarrow \frac{1}{t} \left(\exp\left(1 - \frac{t}{2} + o(t)\right) - e \right)$$

$$= \frac{e}{t} \left(\exp\left(\underbrace{-\frac{t}{2} + o(t)}_{s \rightarrow 0}\right) - 1 \right)$$

$$e^s = 1 + s + o(s) \rightarrow \frac{e}{t} \left(\cancel{1} - \frac{t}{2} + o(t) - \cancel{1} \right)$$

$$= e \left(-\frac{1}{2} + o(1) \right) \rightarrow -\frac{e}{2}.$$

Se avessimo usato l'espansione di ordine più basso $\log(1+t) = t + o(t)$ non si sarebbe potuto concludere:

$$x \left(\left(1 + \frac{1}{x} \right)^x - e \right) = \frac{1}{t} \left(\exp\left(\frac{\log(1+t)}{t}\right) - e \right)$$

$$\log(1+t) = t + o(t) \rightarrow \frac{1}{t} \left(\exp(1 + o(1)) - e \right)$$
$$= \frac{e}{t} \left(\exp(\underbrace{o(1)}_{s \rightarrow 0}) - 1 \right)$$

$$e^s = 1 + s + o(s) \rightarrow \frac{e}{t} \left(\cancel{1} + o(1) - \cancel{1} \right)$$

$$= e \frac{o(1)}{t} \rightarrow ?$$

Non è possibile confrontare gli infinitesimi $o(1)$ e t

ALGEBRA DELL'O-PICCOLO

Per il simbolo dell'O-piccolo valgono le seguenti regole che derivano dalle proprietà dei limiti. Per $x \rightarrow 0$,

- 1) $\forall a \geq 0$ e $\forall c \neq 0$ $c \cdot o(x^a) = o(c \cdot x^a) = o(x^a)$
- 2) $\forall b > a \geq 0$ e $\forall c$ $c \cdot x^b + o(x^a) = o(x^a)$
- 3) $\forall b \geq a \geq 0$ $o(x^b) + o(x^a) = o(x^a)$
- 4) $\forall a \geq -b$ $x^b o(x^a) = o(x^{a+b})$
- 5) $\forall a, b \geq 0$ $o(x^b) \cdot o(x^a) = o(x^{a+b})$
- 6) $\forall a, b \geq 0$ e $\forall c$ $(c \cdot x^a + o(x^a))^b = c^b \cdot x^{ab} + o(x^{ab})$
- 7) $\forall a \geq 0$ e $\forall c$ $o(c \cdot x^a + o(x^a)) = o(x^a)$

Per $x \rightarrow x_0$ basta sostituire $o(x^a)$ con $o((x-x_0)^a)$.

ESEMPI

- $\lim_{x \rightarrow 0} \left(\frac{\sin(x)}{x} \right)^{1/x^2} = ?$

Per $x \rightarrow 0$,

$$\left(\frac{\sin(x)}{x} \right)^{1/x^2} = \exp \left(\frac{1}{x^2} \log \left(\frac{\sin(x)}{x} \right) \right)$$

$$\sin(x) = x - \frac{x^3}{6} + o(x^3) \Rightarrow \exp \left(\frac{1}{x^2} \log \left(1 - \frac{x^2}{6} + o(x^2) \right) \right)$$

$$\begin{aligned} \log(1+t) &= t + o(t) \Rightarrow \exp \left(\frac{1}{x^2} \left(-\frac{x^2}{6} + o(x^2) + o \left(-\frac{x^2}{6} + o(x^2) \right) \right) \right) \\ t &= -\frac{x^2}{6} + o(x^2) \rightarrow 0 \\ &= \exp \left(-\frac{1}{6} + o(1) \right) \rightarrow e^{-\frac{1}{6}} \end{aligned}$$

- $\lim_{x \rightarrow 0} \frac{\sin(x^2) - \sin^2(x)}{x^2(\cos(x^2) - \cos^2(x))} = ?$

Per $x \rightarrow 0$,

$$\frac{\sin(x^2) - \sin^2(x)}{x^2(\cos(x^2) - \cos^2(x))}$$

$$\begin{aligned} \sin(t) &= t - \frac{t^3}{6} + o(t^3) \downarrow \\ \cos(t) &= 1 - \frac{t^2}{2} + o(t^2) \uparrow \end{aligned} \quad \begin{aligned} &= \frac{x^2 - \frac{(x^2)^3}{6} + o(x^6) - \left(x - \frac{x^3}{6} + o(x^3)\right)^2}{x^2\left(1 - \frac{(x^2)^2}{2} + o(x^4) - \left(1 - \frac{x^2}{2} + o(x^2)\right)^2\right)} \\ &= \frac{x^2 - \frac{x^6}{6} + o(x^6) - \left(x^2 - 2\frac{x^4}{6} + o(x^4)\right)}{x^2\left(1 - \frac{x^4}{2} + o(x^4) - \left(1 - 2\frac{x^2}{2} + o(x^2)\right)\right)} \\ &= \frac{\cancel{x^2} - \frac{x^6}{6} + o(x^6) - (\cancel{x^2} - 2\frac{x^4}{6} + o(x^4))}{x^2(\cancel{1} - \frac{x^4}{2} + o(x^4) - (\cancel{1} - 2\frac{x^2}{2} + o(x^2)))} \\ &= \frac{\frac{1}{3}x^4 + o(x^4)}{x^2(x^2 + o(x^2))} = \frac{\frac{1}{3}x^4 + o(x^4)}{x^4 + o(x^4)} \rightarrow \frac{1}{3} \end{aligned}$$

- $\lim_{x \rightarrow 1} \frac{\sqrt{x+3} - \sqrt{3x+5}}{(\log(x))^2} = ?$

Per $x \rightarrow 1$, $t = x - 1 \rightarrow 0$ e

$$\frac{\sqrt{x+3} - \sqrt{3x+5}}{(\log(x))^2} = \frac{\sqrt{4+t} - \sqrt{8+3t}}{(\log(1+t))^2}$$

$$\log(1+t) = t + o(t)$$

$$\downarrow = 2 \frac{\left(1 + \frac{t}{4}\right)^{\frac{1}{2}} - \left(1 + \frac{3t}{8}\right)^{\frac{1}{3}}}{t^2 + o(t^2)}$$

$$(1+s)^{\frac{1}{2}} = 1 + \frac{s}{2} - \frac{s^2}{8} + o(s^2), \quad (1+s)^{\frac{1}{3}} = 1 + \frac{s}{3} - \frac{s^2}{9} + o(s^2)$$

$$\downarrow = \frac{2}{t^2 + o(t^2)} \left(\cancel{1} + \cancel{\frac{t}{8}} - \frac{t^2}{8 \cdot 16} + o(t^2) - \cancel{1} - \cancel{\frac{t}{8}} + \frac{t^2}{64} + o(t^2) \right)$$

$$= \frac{\frac{t^2}{64} + o(t^2)}{t^2 + o(t^2)} \rightarrow \frac{1}{64}$$

$$\bullet \lim_{n \rightarrow \infty} n^2 \left(\exp\left(\frac{1}{n} - \frac{1}{n^2}\right) - \frac{\sqrt{n^2+2}}{n-1} \right) = ?$$

Per $n \rightarrow \infty$,

$$\exp\left(\frac{1}{n} - \frac{1}{n^2}\right) = 1 + \left(\frac{1}{n} - \frac{1}{n^2}\right) + \frac{1}{2} \left(\frac{1}{n} - \frac{1}{n^2}\right)^2 + o\left(\frac{1}{n^2}\right)$$

$$e^x = 1 + x + \frac{x^2}{2} + o(x^2) \quad \uparrow$$

$$= 1 + \frac{1}{n} - \frac{1}{n^2} + \frac{1}{2} \cdot \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{1}{n} - \frac{1}{2} \cdot \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

molte

$$\frac{\sqrt{n^2+2}}{n-1} = \cancel{n} \left(1 + \frac{2}{n^2}\right)^{\frac{1}{2}} \cdot \frac{1}{\cancel{n}} \left(1 - \frac{1}{n}\right)^{-1}$$

$$(1+x)^{\frac{1}{2}} = 1 + \frac{x}{2} + o(x) \quad (1-x)^{-1} = 1 + x + x^2 + o(x^2)$$

$$= \left(1 + \frac{1}{2} \cdot \frac{2}{n^2} + o\left(\frac{1}{n^2}\right)\right) \left(1 + \frac{1}{n} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right)$$

$$= 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{1}{n} + \frac{2}{n^2} + o\left(\frac{1}{n^2}\right).$$

Quindi

$$n^2 \left(\exp\left(\frac{1}{n} - \frac{1}{n^2}\right) - \frac{\sqrt{n^2+2}}{n-1} \right)$$

$$= n^2 \left(\cancel{1} + \cancel{\frac{1}{n}} - \frac{1}{2} \cdot \frac{1}{n^2} + O\left(\frac{1}{n^2}\right) - \left(\cancel{1} + \cancel{\frac{1}{n}} + \frac{2}{n^2} + O\left(\frac{1}{n^2}\right) \right) \right)$$

$$= n^2 \left(-\frac{1}{2} \cdot \frac{1}{n^2} - \frac{2}{n^2} + O\left(\frac{1}{n^2}\right) \right)$$

$$= -\frac{5}{2} + O(1) \rightarrow -\frac{5}{2}.$$