ANALISI MATEMATICA 1 - LEZIONE 21

TECNICHE DI INTEGRAZIONE

1) PER SOSTITUZIONE

Se $F' = f$ allora

$$\int f(g(x))\, g'(x)\, dx \underset{\substack{t=g(x)\\ dt=d(g(x))=g'(x)dx}}{=} \int f(t)\, dt = F(t) + c = F(g(x)) + c.$$

$\overset{S}{\underset{D}{\leftrightarrows}}$ $d(\cdot)$ DIFFERENZIALE

Tale tecnica segue direttamente dalla regola della derivazione di una funzione composta.

ESEMPI

- $$\int tg(x)\,dx = \int \frac{\overset{-(\cos(x))'}{\overset{\shortparallel}{\sin(x)}}}{\cos(x)}\,dx \underset{t=\cos(x),\ dt=-\sin(x)dx}{=} \int \frac{-dt}{t} = -\log|t| + c$$
$$= -\log|\cos(x)| + c$$

- $$\int x\sqrt{2x^2+3}\,dx \underset{t=2x^2+3\ ,\ dt=4x\,dx}{=} \int \frac{1}{4}\, t^{1/2}\,dt = \frac{1}{4}\,\frac{t^{3/2}}{3/2} + c$$
$$= \frac{1}{6}(2x^2+3)^{3/2} + c$$

- $$\int \frac{1}{e^x + e^{-x}}\,dx \underset{t=e^x,\ x=\log(t),\ dx=\frac{dt}{t}}{=} \int \frac{1}{t+t^{-1}}\cdot\frac{dt}{t} = \int \frac{1}{t^2+1}\,dt$$
$$= \operatorname{arctg}(t) + c = \operatorname{arctg}(e^x) + c$$

2) PER PARTI

$$\int f(x)\underbrace{g'(x)dx}_{d(g(x))} = f(x)g(x) - \int g(x)\underbrace{f'(x)dx}_{d(f(x))}$$

Tale tecnica segue direttamente dalla regola della derivazione di un prodotto di funzioni.

$\boxed{\text{ESEMPI}}$

- $\int \log(x)\overbrace{dx}^{d(x)} = \log(x)\cdot x - \int x\,\overbrace{d(\log(x))}^{\frac{dx}{x}}$

$$= \log(x)\cdot x - \int \cancel{x}\frac{dx}{\cancel{x}}$$

$$= \log(x)\cdot x - x + c$$

- $\int \operatorname{arctg}(x)dx = \operatorname{arctg}(x)\cdot x - \int x\,\overbrace{d(\operatorname{arctg}(x))}^{\frac{dx}{1+x^2}}$

$$= \operatorname{arctg}(x)\cdot x - \int \frac{x}{1+x^2}dx$$

$t = 1+x^2$
$dt = 2x\,dx$

$$= \operatorname{arctg}(x)\cdot x - \frac{1}{2}\int \frac{1}{t}dt$$

$$= \operatorname{arctg}(x)\cdot x - \frac{1}{2}\log|t| + c$$

$$= \operatorname{arctg}(x)\cdot x - \frac{1}{2}\log(1+x^2) + c$$

- $\displaystyle\int x e^x dx = \begin{cases} \displaystyle\int e^x d\left(\frac{x^2}{2}\right) = e^x \frac{x^2}{2} - \int \frac{x^2}{2} \overbrace{d(e^x)}^{e^x dx} \quad ? \\ \displaystyle\int x\, d(e^x) = x e^x - \int e^x \overbrace{d(x)}^{dx} \\ \qquad = x e^x - e^x + c \end{cases}$

- $$\int x^2 e^x dx = \int x^2 d(e^x) = x^2 e^x - \int e^x d(x^2)$$
$$= x^2 e^x - 2\int x e^x dx$$
$$= x^2 e^x - 2(x e^x - e^x + c)$$
$$= x^2 e^x - 2x e^x + 2e^x + c$$

il numero che moltiplica c può essere omesso

OSSERVAZIONE

Se P è un polinomio di grado n allora

$$\int P(x) e^x dx = \int P(x)\, d(e^x) = P(x) e^x - \int e^x d(P(x))$$
$$= P(x) e^x - \int P'(x) e^x dx$$
$$= P(x) e^x - \int P'(x)\, d(e^x)$$
$$= P(x) e^x - P'(x) e^x + \int e^x d(P'(x))$$
$$= P(x) e^x - P'(x) e^x + \int P''(x) e^x dx$$
$$\vdots$$
$$= \left(\sum_{k=0}^{n} (-1)^k P^{(k)}(x)\right) e^x + c$$

- $\int \operatorname{arcsen}(x)\,dx = x\operatorname{arcsen}(x) - \int x\, d(\operatorname{arcsen}(x))$

$$= x\operatorname{arcsen}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$$= x\operatorname{arcsen}(x) - \int \frac{x}{\sqrt{1-x^2}}\,dx$$

$t = 1-x^2$
$dt = -2x\,dx$

$$= x\operatorname{arcsen}(x) + \frac{1}{2}\int t^{-1/2}\,dt$$

$$= x\operatorname{arcsen}(x) + \frac{1}{\cancel{2}}\cdot\frac{t^{1/2}}{\cancel{1/2}} + c$$

$$= x\operatorname{arcsen}(x) + \sqrt{1-x^2} + c$$

- $\int \operatorname{sen}(x)\,e^x\,dx = \int \operatorname{sen}(x)\,d(e^x)$

$$= \operatorname{sen}(x)e^x - \int e^x\,\overbrace{d(\operatorname{sen}(x))}^{\cos(x)dx}$$

$$= \operatorname{sen}(x)e^x - \int \cos(x)e^x\,dx$$

$$= \operatorname{sen}(x)e^x - \int \cos(x)\,d(e^x)$$

$$= \operatorname{sen}(x)e^x - \cos(x)e^x + \int e^x\,\overbrace{d(\cos(x))}^{-\operatorname{sen}(x)dx}$$

$$= \operatorname{sen}(x)e^x - \cos(x)e^x - \underbrace{\boxed{\int \operatorname{sen}(x)e^x\,dx}}_{\text{integrale iniziale}}$$

Quindi

$$\int \operatorname{sen}(x)\,e^x\,dx = \frac{1}{2}(\operatorname{sen}(x) - \cos(x))e^x + c$$

$$\bullet \int \sqrt{1-x^2}\,dx = x\sqrt{1-x^2} - \int x\, d\left(\sqrt{1-x^2}\right)$$

$$= x\sqrt{1-x^2} - \int x \frac{-\cancel{2}x}{\cancel{2}\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} - \int \frac{\pm 1 - x^2}{\sqrt{1-x^2}}\,dx$$

$$= x\sqrt{1-x^2} - \int \left(\sqrt{1-x^2} - \frac{1}{\sqrt{1-x^2}}\right)dx$$

$$= x\sqrt{1-x^2} - \underbrace{\boxed{\int \sqrt{1-x^2}\,dx}}_{\text{integrale iniziale}} + \operatorname{arcsen}(x)$$

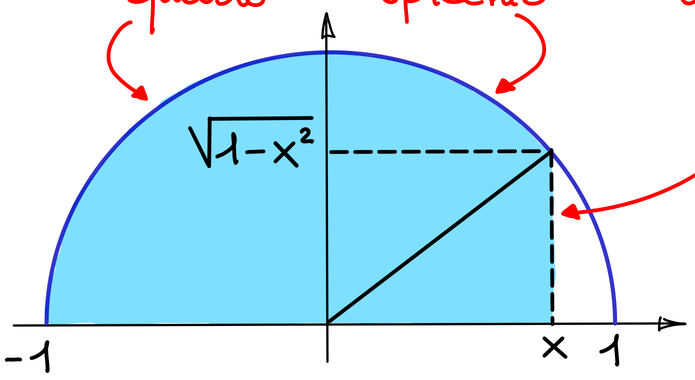
Quindi

$$\int \sqrt{1-x^2}\,dx = \frac{1}{2}\left(x\sqrt{1-x^2} + \operatorname{arcsen}(x)\right) + c.$$

La corrispondente funzione integrale calcola l'area di una parte di un semicerchio di raggio 1:

$$I(x) = \int_{-1}^{x} \sqrt{1-t^2}\,dt = \frac{1}{2}\left[t\sqrt{1-t^2} + \operatorname{arcsen}(t)\right]_{-1}^{x}$$

$$= \underbrace{\frac{\pi}{4}}_{\text{quarto}} + \underbrace{\frac{1}{2}\operatorname{arcsen}(x)}_{\text{spicchio}} + \underbrace{\frac{1}{2}x\sqrt{1-x^2}}_{\text{triangolo}}$$



$$\bullet \int x \log(1+x^4)\,dx = \int \log(1+x^4)\,d\left(\frac{x^2}{2}\right)$$

$$= \frac{x^2}{2}\log(1+x^4) - \int \frac{x^2}{2}\,d\left(\log(1+x^4)\right)$$

$$= \frac{x^2}{2}\log(1+x^4) - \int \frac{x^2}{\cancel{2}} \cdot \frac{\overset{2}{\cancel{4}}x^3}{1+x^4}\,dx$$

$t = x^2$

$dt = 2x\,dx$

$$= \frac{x^2}{2}\log(1+x^4) - \int \frac{t^2 \pm 1}{1+t^2}\,dt$$

$$= \frac{x^2}{2}\log(1+x^4) - \int 1\,dt + \int \frac{1}{1+t^2}\,dt$$

$$= \frac{x^2}{2}\log(1+x^4) - t + \operatorname{arctg}(t) + c$$

$$= \frac{x^2}{2}\log(1+x^4) - x^2 + \operatorname{arctg}(x^2) + c$$