ANALISI MATEMATICA 1 - LEZIONE 17

ESEMPI

- $\lim_{x \to +\infty} x\left(\left(1+\frac{1}{x}\right)^x - e\right) = ?$

Per $x \to +\infty$, $t = \frac{1}{x} \to 0^+$ e

$$x\left(\left(1+\frac{1}{x}\right)^x - e\right) = \frac{(1+t)^{1/t} - e}{t} = \frac{1}{t}\left(\exp\left(\frac{\log(1+t)}{t}\right) - e\right)$$

$\log(1+t) = t - \frac{t^2}{2} + o(t^2)$

$$= \frac{1}{t}\left(\exp\left(1 - \frac{t}{2} + o(t)\right) - e\right)$$

$$= \frac{e}{t}\left(\exp\left(\underbrace{-\frac{t}{2} + o(t)}_{=s \to 0}\right) - 1\right)$$

$e^s = 1 + s + o(s)$

$$= \frac{e}{t}\left(\cancel{1} - \frac{t}{2} + o(t) - \cancel{1}\right)$$

$$= e\left(-\frac{1}{2} + o(1)\right) \to -\frac{e}{2}$$

Se avessimo usato l'espansione di ordine più basso $\log(1+t) = t + o(t)$ non si sarebbe potuto concludere:

$$x\left(\left(1+\frac{1}{x}\right)^x - e\right) = \frac{1}{t}\left(\exp\left(\frac{\log(1+t)}{t}\right) - e\right)$$

$\log(1+t) = t + o(t)$

$$= \frac{1}{t}\left(\exp(1 + o(1)) - e\right)$$

$$= \frac{e}{t}\left(\exp(\underbrace{o(1)}_{=s \to 0}) - 1\right)$$

$e^s = 1 + s + o(s)$

$$= \frac{e}{t}\left(\cancel{1} + o(1) - \cancel{1}\right)$$

$$= e\,\frac{o(1)}{t} \to ?$$

Non è possibile confrontare gli infinitesimi $o(1)$ e t

- $$\lim_{x\to 1} \frac{2\sqrt{x}+\frac{1}{x}-3}{e^x-e(1+\log(x))} = ?$$

1) Applichiamo de L'Hopital:

$$\overset{H}{\underset{\frac{0}{0}}{=}} \lim_{x\to 1} \frac{\frac{1}{\sqrt{x}}-\frac{1}{x^2}}{e^x-\frac{e}{x}} \overset{H}{\underset{\frac{0}{0}}{=}} \lim_{x\to 1} \frac{-\frac{1}{2}x^{-\frac{3}{2}}+2x^{-3}}{e^x+ex^{-2}}$$

$$= \frac{-\frac{1}{2}+2}{e+e+0+0} = \frac{3}{4e}$$

2) Applichiamo gli sviluppi di Taylor:

Per $x\to 1$, $t=x-1\to 0$ e

$$\frac{2\sqrt{x}+\frac{1}{x}-3}{e^x-e(1+\log(x))} = \frac{2(1+t)^{1/2}+(1+t)^{-1}-3}{e^{1+t}-e(1+\log(1+t))}$$

$$(1+t)^{1/2} = 1+\frac{t}{2}-\frac{t^2}{8}+o(t^2), \quad (1+t)^{-1} = 1-t+t^2+o(t^2)$$

$$e^t = 1+t+\frac{t}{2}+o(t^2), \quad \log(1+t) = 1+t-\frac{t^2}{2}+o(t^2)$$

$$= \frac{2\left(1+\frac{t}{2}-\frac{t^2}{8}+o(t^2)\right)+\left(1-t+t^2+o(t^2)\right)-3}{e\left(1+t+\frac{t^2}{2}+o(t^2)\right)-e\left(1+t-\frac{t^2}{2}+o(t^2)\right)}$$

$$= \frac{\left(-\frac{1}{4}+1\right)t^2+o(t^2)}{e\left(\frac{1}{2}+\frac{1}{2}\right)t^2+o(t^2)} \longrightarrow \frac{3}{4e}$$

ALGEBRA DELL'O-PICCOLO

Per il simbolo dell'o-piccolo valgono le seguenti regole che derivano dalle proprietà dei limiti. Per $x \to 0$,

1) $\forall \alpha \geq 0$ e $\forall c \neq 0$ $\qquad c \cdot o(x^\alpha) = o(c \cdot x^\alpha) = o(x^\alpha)$

2) $\forall \beta > \alpha \geq 0$ $\forall c$ $\qquad c \cdot x^\beta + o(x^\alpha) = o(x^\alpha)$

3) $\forall \beta \geq \alpha \geq 0$ $\qquad o(x^\beta) + o(x^\alpha) = o(x^\alpha)$

4) $\forall \alpha \geq -\beta$ $\qquad x^\beta o(x^\alpha) = o(x^{\alpha+\beta})$

5) $\forall \alpha, \beta \geq 0$ $\qquad o(x^\beta) \cdot o(x^\alpha) = o(x^{\alpha+\beta})$

6) $\forall \alpha, \beta \geq 0$ $\forall c$ $\qquad (c \cdot x^\alpha + o(x^\alpha))^\beta = c^\beta x^{\alpha\beta} + o(x^{\alpha\beta})$

7) $\forall \alpha \geq 0$ $\forall c$ $\qquad o(c \cdot x^\alpha + o(x^\alpha)) = o(x^\alpha)$

Per $x \to x_0$ basta sostituire $o(x^\alpha)$ con $o((x-x_0)^\alpha)$.

ESEMPI

- $\lim\limits_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{(tg(x))^2} \right) = ?$

Per $x \to 0$, $\qquad (tg(x))^2 = \left(x + \frac{x^3}{3} + o(x^3)\right)^2 = x^2 + \frac{2x^4}{3} + o(x^4).$

$= x^2 + o(x^2)$

Così

$$\frac{1}{x^2} - \frac{1}{(tg(x))^2} = \frac{(tg(x))^2 - x^2}{x^2 (tg(x))^2} = \frac{\cancel{x^2} + \frac{2x^4}{3} + o(x^4) - \cancel{x^2}}{x^2(x^2 + o(x^2))}$$

$$= \frac{\frac{2x^4}{3} + o(x^4)}{x^4 + o(x^4)} \longrightarrow \frac{2}{3}$$

- $$\lim_{x\to 0} \frac{\operatorname{sen}(x^2)-\operatorname{sen}^2(x)}{x^2(\cos(x^2)-\cos^2(x))} = ?$$

Per $x\to 0$,

$$\frac{\operatorname{sen}(x^2)-\operatorname{sen}^2(x)}{x^2(\cos(x^2)-\cos^2(x))}$$

$\operatorname{sen}(t)=t-\frac{t^3}{6}+o(t^3)$

$\cos(t)=1-\frac{t^2}{2}+o(t^2)$

$$= \frac{x^2-\frac{(x^2)^3}{6}+o(x^6)-\left(x-\frac{x^3}{6}+o(x^3)\right)^2}{x^2\left(1-\frac{(x^2)^2}{2}+o(x^4)-\left(1-\frac{x^2}{2}+o(x^2)\right)^2\right)}$$

$$= \frac{x^2-\frac{x^6}{6}+o(x^6)-\left(x^2-2\frac{x^4}{6}+o(x^4)\right)}{x^2\left(1-\frac{x^4}{2}+o(x^4)-\left(1-2\frac{x^2}{2}+o(x^2)\right)\right)}$$

$$= \frac{\cancel{x^2}-\frac{x^6}{6}+o(x^6)-\left(\cancel{x^2}-2\frac{x^4}{6}+o(x^4)\right)}{x^2\left(\cancel{1}-\frac{x^4}{2}+o(x^4)-\left(\cancel{1}-2\frac{x^2}{2}+o(x^2)\right)\right)}$$

$$= \frac{\frac{1}{3}x^4+o(x^4)}{x^2\left(x^2+o(x^2)\right)} = \frac{\frac{1}{3}x^4+o(x^4)}{x^4+o(x^4)} \to \frac{1}{3}$$

- $$\lim_{x\to 1} \frac{\sqrt{x+3}-\sqrt[3]{3x+5}}{(\log(x))^2} = ?$$

Per $x\to 1$, $t=x-1\to 0$ e

$$\frac{\sqrt{x+3}-\sqrt[3]{3x+5}}{(\log(x))^2} = \frac{\sqrt{4+t}-\sqrt[3]{8+3t}}{(\log(1+t))^2}$$

$\log(1+t) = t + o(t)$

$$= 2\,\frac{\left(1+\frac{t}{4}\right)^{1/2} - \left(1+\frac{3t}{8}\right)^{1/3}}{t^2+o(t^2)}$$

$(1+s)^{1/2} = 1 + \frac{s}{2} - \frac{s^2}{8} + o(s^2)$, $(1+s)^{1/3} = 1 + \frac{s}{3} - \frac{s^2}{9} + o(s^2)$

$$= \frac{2}{t^2+o(t^2)}\left(\cancel{1} + \cancel{\frac{t}{8}} - \underbrace{\frac{t^2}{8\cdot 16}}_{=2\cdot 64} + o(t^2) - \cancel{1} - \cancel{\frac{t}{8}} + \frac{t^2}{64} + o(t^2)\right)$$

$$= \frac{\frac{t^2}{64} + o(t^2)}{t^2+o(t^2)} \longrightarrow \frac{1}{64}$$

- $\lim\limits_{n\to\infty} n^2\left(\exp\left(\frac{1}{n} - \frac{1}{n^2}\right) - \frac{\sqrt{n^2+2}}{n-1}\right) = ?$

Per $n \to \infty$,

$$\exp\left(\frac{1}{n} - \frac{1}{n^2}\right) = 1 + \left(\frac{1}{n} - \frac{1}{n^2}\right) + \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n^2}\right)^2 + o\left(\frac{1}{n^2}\right)$$

$e^x = 1 + x + \frac{x^2}{2} + o(x^2)$

$$= 1 + \frac{1}{n} - \frac{1}{n^2} + \frac{1}{2}\cdot\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{1}{n} - \frac{1}{2}\cdot\frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

Inoltre

$$\frac{\sqrt{n^2+2}}{n-1} = \cancel{n}\left(1+\frac{2}{n^2}\right)^{1/2}\cdot\frac{1}{\cancel{n}}\left(1-\frac{1}{n}\right)^{-1}$$

$(1+x)^{1/2} = 1 + \frac{x}{2} + o(x)$ $\qquad (1-x)^{-1} = 1 + x + x^2 + o(x^2)$

$$= \left(1 + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{n^2} + o\left(\frac{1}{n^2}\right)\right)\left(1 + \frac{1}{n} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)\right)$$

$$= 1 + \frac{1}{n} + \frac{1}{n^2} + \frac{1}{n^2} + o\left(\frac{1}{n^2}\right)$$

$$= 1 + \frac{1}{n} + \frac{2}{n^2} + o\left(\frac{1}{n^2}\right).$$

Quindi

$$n^2\left(\exp\left(\frac{1}{n}-\frac{1}{n^2}\right)-\frac{\sqrt{n^2+2}}{n-1}\right)$$

$$=n^2\left(\cancel{1}+\cancel{\frac{1}{n}}-\frac{1}{2}\cdot\frac{1}{n^2}+o\left(\frac{1}{n^2}\right)-\left(\cancel{1}+\cancel{\frac{1}{n}}+\frac{2}{n^2}+o\left(\frac{1}{n^2}\right)\right)\right)$$

$$=n^2\left(-\frac{1}{2}\cdot\frac{1}{n^2}-\frac{2}{n^2}+o\left(\frac{1}{n^2}\right)\right)$$

$$=-\frac{5}{2}+o(1)\longrightarrow-\frac{5}{2}$$