1. Per ciascuna successione di funzioni $\{f_n\}_{n\geq 1}$, determinare il limite puntuale negli intervalli indicati e verificare se la convergenza è anche uniforme.

(a) $f_n(x) = \dfrac{(n+1)x}{1+n^2x^2}$ in $[0,+\infty)$,

(b) $f_n(x) = nx^2e^{-nx}$ in $[0,+\infty)$,

(c) $f_n(x) = \left(1+\dfrac{x}{n}\right)^n$ in $[0,1]$,

(d) $f_n(x) = \sqrt[n]{1+|x|^n}$ in $\mathbb{R}$,

(e) $f_n(x) = n\left(\sqrt{x+\dfrac{1}{n}}-\sqrt{x}\right)$ in $(0,1)$,

(f) $f_n(x) = (n^2x+1)e^{-nx^2}$ in $[-1,1]$.

Svolgimento:

(a) $\forall x \in [0,+\infty)$, $\dfrac{(n+1)x}{1+n^2x^2} \xrightarrow{n\to\infty} 0 = f(x)$.

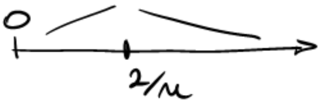
$$f_n'(x) = \frac{n+1}{(1+n^2x^2)^2}\cdot\Big(\underbrace{1+n^2x^2 - x\cdot n^2\cdot 2x}_{(1-nx)(1+nx)}\Big)$$

$$\sup_{[0,+\infty)} |f_n(x)-f(x)| = f_n\left(\tfrac{1}{n}\right) = \frac{(n+1)\cdot\frac{1}{n}}{1+1} \to \frac{1}{2}$$

Quindi $f_n \overset{u}{\not\to} f$ in $[0,+\infty)$. $\square$

(b) $\forall x \in [0,+\infty)$, $nx^2\cdot e^{-nx} \xrightarrow{n\to\infty} 0 = f(x)$.

$$f_n'(x) = n\left(2x e^{-nx} + x^2(-ne^{-nx})\right) = x e^{-nx}(2-nx)$$

così  .

$$\sup_{[0,+\infty)} |f(x)-f_n(x)| = f\left(\tfrac{2}{n}\right) = \frac{4}{n}e^{-2} \to 0$$

Quindi $f_n \xrightarrow{u} f$ in $[0,+\infty)$. $\square$

(c) $\forall x \in [0,1]$, $\left(1+\dfrac{x}{n}\right)^n \xrightarrow{n\to\infty} e^x = f(x)$.

Osserviamo che $0 \leq f(x) - f_n(x) \overset{d}{=} h_n(x)$ e

$$h_n'(x) = e^x - \left(1+\frac{x}{n}\right)^{n-1} \geq e^x - \left(1+\frac{x}{n}\right)^n \geq 0$$

per $x\geq 0$. Così h_n è crescente in $[0,1]$ e

$$\sup_{x\in[0,1]} |f(x)-f_n(x)| \leq h_n(1) = e - \left(1+\frac{1}{n}\right)^n \to 0$$

$\square$

Quindi $f_n \xrightarrow{u} f$ in $[0,1]$.

(d) $\forall x\in\mathbb{R}$, $\sqrt[n]{1+|x|^n} \longrightarrow \begin{cases} 1 & \text{se } |x|\leq 1 \\ |x| & \text{se } |x|>1 \end{cases} = f(x)$

Notiamo che

$$(1+t)^\alpha \leq 1+\alpha t \quad \text{per } \alpha\in(0,1] \text{ e per } t\geq 0. \quad (*)$$

Per $|x|\leq 1$,

$$0\leq f_n(x)-f(x) = (1+|x|^n)^{\frac{1}{n}}-1 \underset{(*)}{\leq} \cancel{1}+\frac{|x|^n}{n}-\cancel{1} \leq \frac{1}{n}.$$

Per $|x|\geq 1$,

$$0\leq f_n(x)-f(x) = |x|\left(\frac{1}{|x|^n}+1\right)^{\frac{1}{n}}-|x|$$

$$\underset{(*)}{\leq} |x|\left(\cancel{1}+\frac{1}{n|x|^n}-\cancel{1}\right) = \frac{1}{n|x|^{n-1}} \leq \frac{1}{n}.$$

Quindi $\sup\limits_{x\in\mathbb{R}} |f_n(x)-f(x)| \leq \frac{1}{n} \to 0$ e così $f_n \xrightarrow{u} f$ in $\mathbb{R}$. $\square$

(e) $\forall x\in(0,1)$, $\dfrac{\sqrt{x+\frac{1}{n}}-\sqrt{x}}{1/n} \xrightarrow{n\to\infty} \dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{2\sqrt{x}} = f(x)$.

Ma per $x=\frac{1}{n}\in(0,1)$

$$f\left(\tfrac{1}{n}\right)-f_n\left(\tfrac{1}{n}\right) = \frac{\sqrt{n}}{2} - n\cdot\left(\frac{\sqrt{2}}{\sqrt{n}}-\frac{1}{\sqrt{n}}\right) = \sqrt{n}\left(\frac{3}{2}-\sqrt{2}\right) \to +\infty$$

Quindi $f_n \not\xrightarrow{u} f$ in $(0,1)$. $\square$

(f) $\forall x\in[-1,1]$, $(n^2x+1)e^{-nx^2} \to \begin{cases} 1 & \text{se } x=0 \\ 0 & \text{se } 0<|x|\leq 1 \end{cases} = f(x)$

Dato che la funzione limite f non è continua si ha che $f_n \not\xrightarrow{u} f$ in $[-1,1]$. $\square$

2. Dimostrare le seguenti affermazioni.

(a) Per $x < 1$ e $n \in \mathbb{N}^+$, $\quad \ln\left(\dfrac{1}{1-x}\right) = \displaystyle\sum_{k=1}^{n} \frac{x^k}{k} + \int_0^x \frac{t^n}{1-t}\, dt$.

(b) Per $x \in (-1, 1]$, $\quad \ln(1+x) = \displaystyle\sum_{k=1}^{\infty} \frac{(-1)^{k+1} x^k}{k}$.

Svolgimento:

(a) Sia $F(x) = \displaystyle\int_0^x \frac{t^n}{1-t}\, dt$ allora basta verificare che $G(x) = \ln\left(\dfrac{1}{1-x}\right) - \displaystyle\sum_{k=1}^{n} \frac{x^k}{k}$ è la primitiva di $\dfrac{x^n}{1-x}$ tale che $G(0) = 0$.

Infatti $G(0) = \ln(1) - \displaystyle\sum_{k=1}^{n} \frac{0^k}{k} = 0$ e per $x < 1$,

$$G'(x) = (1-x)\cdot\left(+\frac{1}{(1-x)^2}\right) - \sum_{k=1}^{n} \cancel{k} \cdot \frac{x^{k-1}}{\cancel{k}} = \frac{1}{1-x} - \sum_{k=0}^{n-1} x^k$$

$$= \frac{1}{1-x} - \frac{1-x^n}{1-x} = \frac{x^n}{1-x}.$$

□

(b) Da (a), per $x > -1$,

$$\ln(1+x) = -\ln\left(\frac{1}{1-(-x)}\right) = \sum_{k=1}^{n} \frac{(-1)^{k+1} x^k}{k} + \int_0^{-x} \frac{t^n}{1-t}\, dt .$$

Così se $x \in (-1, 1]$ allora

$$\left| \int_0^{-x} \frac{t^n}{1-t}\, dt \right| \leq M \left| \int_0^{-x} t^n\, dt \right| \leq M \cdot \frac{|x|^{n+1}}{n+1} \xrightarrow{n\to\infty} 0 .$$

dove $M = \displaystyle\max_{[0,-x]} \left| \frac{1}{1-t} \right| = \max\left(1, \frac{1}{1+x}\right)$ e dunque

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k}$$

□

3. Calcolare i seguenti limiti.

(a) $\displaystyle\lim_{n\to\infty}\int_0^1\left(1+\frac{x}{n}\right)^n dx,$ (b) $\displaystyle\lim_{n\to\infty}\int_1^{+\infty}\frac{n^3(\ln(x))^2}{x^n}\,dx,$

(c) $\displaystyle\lim_{n\to\infty}\int_0^{+\infty}\frac{n(1-\cos(x))}{n^2x^2+1}\,dx,$ (d) $\displaystyle\lim_{n\to\infty}\int_0^1\frac{4+nx^n}{4-x^2}\,dx.$

Svolgimento:

(a) Da 1(c) si ha che $\left(1+\frac{x}{n}\right)^n \longrightarrow e^x$ unif. in $[0,1]$

e quindi
$$\lim_{n\to+\infty}\int_0^1\left(1+\frac{x}{n}\right)^n dx=\int_0^1 e^x dx=e-1.$$
□

(b) Abbiamo che per $n>1$

$$n^3\int_1^{+\infty}\frac{\ln^2(x)}{x^n}\,dx \underset{\substack{\ln x=t\\ \frac{dx}{x}=dt}}{=} n^3\int_0^{+\infty}t^2e^{-(n-1)t}\,dt$$

$$\underset{\text{Integrazione per parti}}{=} n^3\left[\frac{-e^{-(n-1)t}}{(n-1)^3}\left(2+2(n-1)t+(n-1)^2t^2\right)\right]_0^{+\infty}$$

$$=\frac{2n^3}{(n-1)^3}\xrightarrow{n\to+\infty}2.$$
□

(c) Notiamo che per $x\geq 0$

$$0\leq\frac{n(1-\cos(x))}{n^2x^2+1}\underset{(*)}{\leq}\frac{n\left(\frac{x^2}{2}\right)}{n^2x^2+1}\leq\frac{1}{2n}$$

$(*)$ $1-\cos x\leq\frac{x^2}{2}$

ossia $\frac{n(1-\cos x)}{n^2x^2+1}$ converge uniformemente a 0 in $[0,+\infty)$.

Inoltre se $R>0$

$$0\leq\int_R^{+\infty}\frac{n(1-\cos(x))}{n^2x^2+1}\,dx\leq 2n\int_R^{+\infty}\frac{dx}{n^2x^2+1}=2\left[\operatorname{arctg}(nx)\right]_R^{+\infty}$$
$$=\pi-2\operatorname{arctg}(nR).$$

Così

$$0 \le \int_0^{+\infty} \frac{n(1-\cos(x))}{n^2x^2+1}\,dx = \int_0^R \cdots + \int_R^{+\infty} \cdots \le \frac{R}{2n} + \pi - 2\operatorname{arctg}(nR)$$

$$\underset{R=1}{\le} \underbrace{\frac{1}{2n} + \pi - 2\operatorname{arctg}(n)}_{\to 0 \text{ per } n\to\infty}$$

Quindi

$$\lim_{n\to\infty} \int_0^{+\infty} \frac{n(1-\cos(x))}{n^2x^2+1}\,dx = 0.$$

(*) Se $f(x) = \cos x - 1 + \frac{x^2}{2}$ allora $f'(x) = -\operatorname{sen} x + x$ e $f''(x) = -\cos x + 1 \ge 0$. Quindi $f'(x)$ è crescente e $f'(x) \ge f'(0) = 0$ per $x \ge 0$. Infine $f(x)$ è crescente e $f(x) \ge f(0) = 0$. $\square$

(d) Abbiamo che

$$\int_0^1 \frac{4}{4-x^2}\,dx = \int_0^1 \left(\frac{1}{x+2} - \frac{1}{x-2}\right)dx = \left[\ln\left|\frac{x+2}{x-2}\right|\right]_0^1 = \ln 3.$$

Inoltre

$$\int_0^1 \frac{n x^n}{4-x^2}\,dx = \frac{n}{n+1}\int_0^1 \frac{d(x^{n+1})}{4-x^2}$$

$$= \frac{n}{n+1}\left[\frac{x^{n+1}}{4-x^2}\right]_0^1 - \frac{n}{n+1}\int_0^1 \frac{x^{n+1}\cdot 2x}{(4-x^2)^2}\,dx$$

$$= \frac{n}{n+1}\cdot\frac{1}{3} - \frac{2n}{n+1}\int_0^1 \frac{x^{n+2}}{(4-x^2)^2}\,dx \xrightarrow{n\to\infty} \frac{1}{3}$$

perché

$$0 \le \int_0^1 \frac{x^{n+2}}{(4-x^2)^2}\,dx \le \frac{1}{9}\int_0^1 x^{n+2}\,dx = \frac{1}{9(n+3)} \xrightarrow{n\to\infty} 0.$$

Così

$$\lim_{n\to\infty} \int_0^1 \frac{4+nx^n}{4-x^2}\,dx = \ln 3 + \frac{1}{3}.$$

$\square$

4. Sia $\{f_n\}_{n\geq 1}$ una successione di funzioni in $C([0,+\infty))$ che converge uniformemente in $[0,+\infty)$ alla funzione f. Inoltre per ogni $x \geq 0$ e per ogni $n \geq 1$ si ha che $0 < f_n(x) \leq 1$ e $f(x) > 0$. Dimostrare o confutare le seguenti affermazioni.

(a) La successione $\{f_n^2\}_{n\geq 1}$ converge uniformemente in $[0,+\infty)$ a f^2.

(b) La successione $\{1/f_n\}_{n\geq 1}$ converge uniformemente in $[0,+\infty)$ a $1/f$.

(c) $\displaystyle\lim_{n\to\infty}\int_0^{+\infty} f_n(x)dx = \int_0^{+\infty} f(x)dx$.

Svolgimento:

(a) VERO. Per $x \geq 0$, $f(x) \leq 1$ e

$$|f_n^2(x) - f^2(x)| = |f_n(x) + f(x)| \cdot |f_n(x) - f(x)| \leq (1+1)\,|f_n(x) - f(x)|.$$

Quindi

$$\sup_{x\geq 0} |f_n^2(x) - f^2(x)| \leq 2 \cdot \sup_{x \geq 0} |f_n(x) - f(x)| \to 0$$

e possiamo concludere che $f_n^2 \xrightarrow{u} f^2$ in $[0,+\infty)$.

$\square$

(b) FALSO. Ad esempio sia $f_n(x) = \dfrac{n}{(n+1)(x+1)}$ per $n \geq 1$.

Allora $0 < f(x) \leq 1$ e $f_n(x) \xrightarrow{u} f(x) = \dfrac{1}{x+1}$ in $[0,+\infty)$

perché $0 \leq f(x) - f_n(x) = \dfrac{1}{x+1} \cdot \dfrac{1}{n+1} \leq \dfrac{1}{n+1} \to 0$.

Inoltre $\dfrac{1}{f_n(x)} = \dfrac{n(x+1)}{n+1} \to \dfrac{1}{f(x)} = x+1 \quad \forall x \geq 0$

ma $\displaystyle\sup_{x\geq 0}\left|\frac{1}{f_n(x)} - \frac{1}{f(x)}\right| = \sup_{x \geq 0} \frac{x+1}{n} = +\infty$ e così

$\dfrac{1}{f_n} \not\xrightarrow{u} \dfrac{1}{f}$ in $[0,+\infty)$.

$\square$

(c) FALSO. Ad esempio se

$$h_n(x) = \begin{cases} \dfrac{1}{n} & \text{se } x \in [0,n] \\[2mm] \dfrac{e^{-x+n}}{n} & \text{se } x \in [n,+\infty) \end{cases}$$

Allora $f_n(x) \overset{d}{=} \frac{1}{2}\left(h_n(x) + e^{-x}\right)$ è tale che

$0 < f_n(x) \leq 1$ per $x \geq 0$ e $f_n \xrightarrow{u} f = \frac{e^{-x}}{2}$ in $[0,+\infty)$

perché $0 \leq f_n(x) - f(x) = \frac{h_n(x)}{2} \leq \frac{1}{2n} \to 0$.

Inoltre

$$\int_0^{+\infty} f_n(x)\,dx - \int_0^{+\infty} f(x)\,dx = \frac{1}{2}\int_0^{+\infty} h_n(x)\,dx$$

$$= \frac{1}{2n}\left(n + \int_n^{+\infty} e^{-x+n}\,dx\right) = \frac{n+1}{2n} \xrightarrow{n\to\infty} \frac{1}{2} \neq 0. \quad \square$$