1. Calcolare i seguenti integrali indefiniti.

(a) $\displaystyle\int \frac{1}{\sqrt{e^x-1}}\,dx$; (b) $\displaystyle\int \frac{\sqrt[3]{x}-1}{\sqrt{x}+\sqrt[3]{x}}\,dx$;

(c) $\displaystyle\int \frac{1+\sin(x)}{1+\cos(x)}\,dx$; (d) $\displaystyle\int \frac{2e^{4x}}{e^{2x}+1-2\cosh(x)}\,dx$;

(e) $\displaystyle\int \frac{\arctan(x)}{(x+1)^2}\,dx$; (f) $\displaystyle\int \frac{\ln(x)}{\sqrt{x}(1-\sqrt{x})^{3/2}}\,dx$.

Svolgimento:

(a) $$\int \frac{1}{\sqrt{e^x-1}}\,dx \underset{\uparrow}{=} \int \frac{1}{\cancel{t}}\cdot\frac{2\cancel{t}}{t^2+1}\,dt = 2\,\mathrm{arctg}\left(\sqrt{e^x-1}\right)+c$$

$t=\sqrt{e^x-1}$

$\ln(t^2+1)=x$

$\frac{2t}{t^2+1}\,dt=dx$

$\square$

(b) $$\int \frac{\sqrt[3]{x}-1}{\sqrt{x}+\sqrt[3]{x}}\,dx \underset{\uparrow}{=} \int \frac{t^2-1}{t^3+t^2}\,6t^5\,dt = 6\int (t^4-t^3)\,dt$$

$\sqrt[6]{x}=t$

$x=t^6$

$dx=6t^5\,dt$

$$= 6\left(\frac{t^5}{5}-\frac{t^4}{4}\right)+c = \frac{6}{5}x^{5/6}-\frac{3}{2}x^{2/3}+c$$ $\square$

(c) $$\int \frac{1+\operatorname{sen}x}{1+\cos x}\,dx \underset{\uparrow}{=} \int \frac{1+\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}}\,\frac{2\,dt}{1+t^2}$$

$t=\mathrm{tg}\,\frac{x}{2}$

$$= \int \frac{1+t^2+2t}{1+t^2}\,dt = \int\left(1+\frac{2t}{1+t^2}\right)dt$$

$$= t+\ln(1+t^2)+c = \mathrm{tg}\,\frac{x}{2} - \ln\left(\left|\cos\frac{x}{2}\right|\right)+c$$ $\square$

(d) $$\int \frac{2e^{4x}}{e^{2x}+1-2\cosh x}\,dx \underset{t=e^x}{\overset{\uparrow}{=}} \int \frac{2t^4}{t^2+1-(t+1/t)}\,\frac{dt}{t}$$

$$= \int \frac{2t^4}{\underbrace{t^3-t^2+t-1}_{(t^2+1)(t-1)}}\,dt = t^2+2t+\ln|t-1|-\frac{1}{2}\ln(1+t^2) - \operatorname{arctg}(t)+c$$

$$\frac{2t^4}{(t^2+1)(t-1)} = 2t+2+\frac{A}{t-1}+\frac{Bt+C}{t^2+1} \quad \text{con} \begin{cases} A=1 \\ B=C=-1 \end{cases}$$

$$= e^{2x}+2e^x+\ln\left(\frac{|e^x-1|}{\sqrt{1+e^{2x}}}\right)-\operatorname{arctg}(e^x)+c \qquad \square$$

(e) $$\int \frac{\operatorname{arctg}(x)}{(x+1)^2}\,dx = \int \operatorname{arctg} x \, d\left(-\frac{1}{x+1}\right)$$

$$= -\frac{\operatorname{arctg} x}{x+1} + \int \frac{dx}{(x+1)(x^2+1)}$$

$$\frac{1}{(x+1)(x^2+1)} = \frac{A}{x+1}+\frac{Bx+C}{x^2+1} \quad \text{con} \begin{cases} A=C=\frac{1}{2} \\ B=-\frac{1}{2} \end{cases}$$

$$= -\frac{\operatorname{arctg} x}{x+1}+\frac{1}{2}\ln\frac{|x+1|}{\sqrt{x^2+1}}+\frac{1}{2}\operatorname{arctg} x + c \qquad \square$$

(f) $$\int \frac{\ln x}{\sqrt{x}\,(1-\sqrt{x})^{3/2}}\,dx \underset{t=\sqrt{x}}{=} 4\int \frac{\ln t}{(1-t)^{3/2}}\,dt$$

$t^2=x$, $2t\,dt=dx$

$$= 8\int \ln t \, d\left((1-t)^{-\frac{1}{2}}\right) = \frac{8\ln t}{\sqrt{1-t}} - 8\int \frac{dt}{t\sqrt{1-t}}$$

$$= \frac{4\ln x}{\sqrt{1-\sqrt{x}}} + 8\ln\left(\frac{1+\sqrt{1-\sqrt{x}}}{1-\sqrt{1-\sqrt{x}}}\right)+c$$

dove

$$-\int \frac{dt}{t\sqrt{1-t}} \overset{s=\sqrt{1-t}}{\overset{\downarrow}{=}} \int \frac{2\cancel{s}\,ds}{\cancel{s}(1-s^2)} = \int \frac{ds}{1-s}+\int\frac{ds}{1+s} = \ln\left|\frac{1+s}{1-s}\right|+c$$

$\square$

2. Calcolare i seguenti integrali definiti.

(a) $\displaystyle\int_1^e \frac{\ln(1+\ln^2(x))}{x}\,dx$; (b) $\displaystyle\int_{-8}^{10} \sqrt{1+\sqrt{|x-1|}}\,dx$;

(c) $\displaystyle\int_{-1}^{3} \frac{x^2+x-1}{(x^2+3)^2}\,dx$; (d) $\displaystyle\int_{-\pi/6}^{\pi/6} \frac{6+xe^{x^2}}{\cos^3(x)}\,dx$;

(e) $\displaystyle\int_{\pi/6}^{\pi/2} \ln^2(e\sin(x))\cos(x)\,dx$; (f) $\displaystyle\int_{1/4}^{4} \frac{dx}{1+\lfloor 1/x\rfloor}$.

Svolgimento:

$$(a)\ \int_1^e \frac{\ln(1+\ln^2 x)}{x}\,dx \underset{\substack{t=\ln x\\ dt=\frac{dx}{x}}}{=} \int_0^1 \ln(1+t^2)\,dt$$

$$= \Big[\ln(1+t^2)\cdot t\Big]_0^1 - \int_0^1 t\cdot d\big(\ln(1+t^2)\big)$$

$$= \ln 2 - \int_0^1 \frac{t\cdot 2t}{1+t^2}\,dt = \ln 2 - 2\int_0^1\left(1-\frac{1}{1+t^2}\right)dt$$

$$= \ln 2 - 2\Big[t-\operatorname{arctg} t\Big]_0^1 = \ln 2 - 2 + \frac{\pi}{2}.$$ □

$$(b)\ \int_{-8}^{10} \sqrt{1+\sqrt{|x-1|}}\,dx \underset{\substack{t=x-1\\ dt=dx}}{=} \int_{-9}^{9} \underbrace{\sqrt{1+\sqrt{|t|}}}_{\text{pari}}\,dt = 2\int_0^9 \sqrt{1+\sqrt{t}}\,dt$$

$$\underset{\substack{\sqrt{t}=s,\ t=s^2\\ dt=2s\,ds}}{=} 2\int_0^3 \sqrt{1+s}\cdot 2s\,ds \underset{\substack{u=\sqrt{1+s},\ u^2=1+s\\ 2u\,du=ds}}{=} 4\int_1^2 u\,(u^2-1)\cdot 2u\,du$$

$$= 8\left[\frac{u^5}{5}-\frac{u^3}{3}\right]_1^2 = \frac{464}{15}.$$ □

(c) Abbiamo che

$$\frac{x^2+x-1}{(x^2+3)^2} = \frac{Ax+B}{x^2+3} + \frac{Cx+D}{(x^2+3)^2} = \frac{(Ax+B)(x^2+3)+Cx+D}{(x+3)^2}$$

Quindi

$$\begin{cases} A=0 \\ B=1 \\ 3A+C=1 \\ 3B+D=-1 \end{cases} \Rightarrow \begin{cases} A=0 \\ B=1 \\ C=1 \\ D=-4 \end{cases}$$

Così

$$\int_{-1}^{3} \frac{dx}{x^2+3} = \frac{\sqrt{3}}{3}\int_{-1}^{3} \frac{d(x/\sqrt{3})}{1+\left(\frac{x}{\sqrt{3}}\right)^2} = \frac{\sqrt{3}}{3}\left[\operatorname{arctg}\left(\frac{x}{\sqrt{3}}\right)\right]_{-1}^{3}$$

$$= \frac{\sqrt{3}}{3}\left(\frac{\pi}{3}-\left(-\frac{\pi}{6}\right)\right) = \frac{\pi\sqrt{3}}{6},$$

$$\int_{-1}^{3} \frac{x}{(x^2+3)^2}dx = \frac{1}{2}\int_{-1}^{3} \frac{d(x^2+3)}{(x^2+3)^2} = -\frac{1}{2}\left[\frac{1}{x^2+3}\right]_{-1}^{3} = \frac{1}{12}.$$

$$\int_{-1}^{3} \frac{dx}{(x^2+3)^2} = \frac{1}{3}\int_{-1}^{3} \frac{3+x^2-x^2}{(x^2+3)^2}dx = \frac{1}{3}\int_{-1}^{3} \frac{dx}{x^2+3} + \frac{1}{6}\int_{-1}^{3} x\, d\left(\frac{1}{x^2+3}\right)$$

$$= \frac{1}{3}\int_{-1}^{3}\frac{dx}{x^2+3} + \frac{1}{6}\left[\frac{x}{x^2+3}\right]_{-1}^{3} - \frac{1}{6}\int_{-1}^{3}\frac{dx}{x^2+3}$$

$$= \frac{1}{6}\int_{-1}^{3}\frac{dx}{x^2+3} + \frac{1}{6}\left(\frac{3}{12}+\frac{1}{4}\right) = \frac{\pi\sqrt{3}}{36} + \frac{1}{12}.$$

Infine

$$\int_{-1}^{3} \frac{x^2+x-1}{(x^2+3)^2}dx = \frac{\pi\sqrt{3}}{6} + \frac{1}{12} - \frac{\pi\sqrt{3}}{9} - \frac{1}{3} = \frac{\pi\sqrt{3}}{18} - \frac{1}{4}.$$

$\square$

(d) Osserviamo che

$\dfrac{6}{\cos^3(x)}$ è continua e pari in $\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$,

$\dfrac{x e^{x^2}}{\cos^3(x)}$ è continua e dispari in $\left[-\frac{\pi}{6},\frac{\pi}{6}\right]$.

Quindi l'integrale richiesto è

$$I = 12\int_0^{\pi/6} \frac{dx}{\cos^3(x)} = 12\int_0^{\pi/6} \frac{\cos x}{\cos^4(x)}\,dx$$

$$= 12\int_0^{\pi/6} \frac{d(\operatorname{sen} x)}{(1-\operatorname{sen}^2 x)^2} = 12\int_0^{1/2} \frac{dt}{(1-t^2)^2}.$$

ora

$$\frac{1}{(1-t^2)^2} = \frac{A}{t-1} + \frac{B}{(t-1)^2} + \frac{C}{t+1} + \frac{D}{(t+1)^2}$$

$$\Rightarrow B = D = \frac{1}{4}, \quad C = \frac{1}{4}, \quad A = -\frac{1}{4}$$

così

$$I = \frac{12}{4}\left[-\ln|t-1| + \ln|t+1| - \frac{1}{t-1} - \frac{1}{t+1}\right]_0^{1/2}$$

$$= 3\ln 3 + 4.$$

□

(e) $$\int_{\frac{\pi}{6}}^{\pi/2} \ln^2(e\operatorname{sen} x)\cos x\,dx \underset{\substack{t = e\operatorname{sen} x \\ \frac{dt}{e} = \cos x\,dx}}{=} \frac{1}{e}\int_{\frac{e}{2}}^{e} \ln^2(t)\,dt$$

$$\underset{\substack{s = \ln t \\ e^s ds = dt}}{=} \frac{1}{e}\int_{1-\ln 2}^{1} s^2 e^s\,ds = \frac{1}{e}\left[e^s\,(\overbrace{s^2-2s+2}^{(s-1)^2+1})\right]_{1-\ln 2}^{1}$$

$$= 1 - \frac{1}{2}(\ln^2 2 + 1) = \frac{1}{2}(1-\ln^2 2).$$

□

(f) Abbiamo che:

$$\left\lfloor 1/x \right\rfloor = 0 \quad \text{se } x \in (1, +\infty),$$

$$\left\lfloor 1/x \right\rfloor = n \quad \text{se} \quad x \in \left(\frac{1}{n+1}, \frac{1}{n}\right] \quad \text{per } n \in \mathbb{N}^+.$$

Quindi la funzione $\dfrac{1}{1+\lfloor 1/x \rfloor}$ è costante a tratti e

$$\int_{1/4}^{4} \frac{dx}{1+\lfloor 1/x \rfloor} = \sum_{n=1}^{3} \int_{\frac{1}{n+1}}^{\frac{1}{n}} \frac{dx}{1+n} + \int_{1}^{4} \frac{dx}{1+0}$$

$$= \sum_{n=1}^{3} \frac{1}{1+n} \cdot \left(\frac{1}{n} - \frac{1}{n+1}\right) + (4-1)$$

$$= \sum_{n=1}^{3} \frac{1}{n(n+1)^2} + 3 = \frac{1}{4} + \frac{1}{18} + \frac{1}{48} + 3 = \frac{479}{144}. \quad \square$$

3. Dimostrare le seguenti affermazioni.

(a) Se f è continua in $[a,b]$ tale che

$$\int_a^b |f(x)|\,dx = 0,$$

allora f è identicamente nulla in $[a,b]$.

(b) Se f è convessa in $[a,b]$ allora è integrabile in $[a,b]$ e

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Svolgimento:

(a) Se per assurdo f non fosse identicamente nulla $\exists\, x_0 \in [a,b] : |f(x_0)| > 0$. Per la continuità di f anche $|f|$ è continua e per $\varepsilon = \frac{|f(x_0)|}{2} > 0$ $\exists\, c, d$: con $c < d$ e $a \leq c \leq x_0 \leq d \leq b$ tali che

$$\big|\,|f(x)| - |f(x_0)|\,\big| \leq \varepsilon \implies |f(x)| \geq |f(x_0)| - \varepsilon = \frac{|f(x_0)|}{2}.$$

così per la monotonia dell'integrale

$$0 = \int_a^b |f(x)|\,dx = \underbrace{\int_a^c |f(x)|\,dx}_{\geq 0} + \int_c^d |f(x)|\,dx + \underbrace{\int_d^b |f(x)|\,dx}_{\geq 0}$$

$$\geq \int_c^d |f(x)|\,dx \geq \int_c^d \frac{|f(x_0)|}{2}\,dx = \frac{|f(x_0)|}{2}(d-c) > 0$$

Contraddizione!! $\square$

(b) La convessità in $[a,b]$ implica la continuità in (a,b) e la limitatezza in $[a,b]$. Quindi f è integrabile in $[a,b]$.

Per la convessità, $\forall x \in [a,b]$,

$$f(x) \leq \frac{f(b)-f(a)}{b-a}(x-a)+f(a)$$

e integrando in $[a,b]$ si ottiene

$$\int_a^b f(x)\,dx \leq \frac{f(b)-f(a)}{b-a}\int_a^b (x-a)\,dx + f(a)\int_a^b dx$$

$$= \frac{f(b)-f(a)}{b-a}\cdot\frac{(b-a)^2}{2}+f(a)(b-a)$$

$$= (b-a)\cdot\frac{f(a)+f(b)}{2}.$$

Ancora per la convessità, $\forall t \in \left[0, \frac{b-a}{2}\right]$,

$$f\left(\frac{a+b}{2}\right) \leq \frac{f\left(\frac{a+b}{2}+t\right)+f\left(\frac{a+b}{2}-t\right)}{2}$$

e integrando in $\left[0,\frac{b-a}{2}\right]$ si ottiene

$$f\left(\frac{a+b}{2}\right)\cdot\frac{b-a}{2} \leq \frac{1}{2}\int_0^{\frac{b-a}{2}} f\left(\frac{a+b}{2}+t\right)dt + \frac{1}{2}\int_0^{\frac{b-a}{2}} f\left(\frac{a+b}{2}-t\right)dt$$

$$= \frac{1}{2}\int_{\frac{a+b}{2}}^{b} f(s)\,ds + \frac{1}{2}\int_a^{\frac{a+b}{2}} f(s)\,ds = \frac{1}{2}\int_a^b f(s)\,ds.$$

Così

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(s)\,ds.$$

$\square$

4. Sia $F(t) = \displaystyle\int_t^{t^2} e^{-x^2}\, dx$. Dimostrare o confutare le seguenti proposizioni.

(a) La derivata prima di F si annulla almeno due volte in $\mathbb{R}$.

(b) F ammette un punto di minimo assoluto in $\mathbb{R}$.

(c) F ammette un punto di massimo assoluto in $\mathbb{R}$.

(d) Esiste $a > 0$ tale che F è strettamente convessa in $[a, +\infty)$.

(e) $\dfrac{5}{4} < F(-1) < \dfrac{5}{3}$.

Svolgimento:

(a) VERO. Per il teo. fondamentale del calcolo integrale la funzione $F(t)$ è derivabile e

$$F'(t) = \frac{d}{dt}\left(\int_0^{t^2} e^{-x^2} dx\right) - \frac{d}{dt}\left(\int_0^{t} e^{-x^2} dx\right)$$

$$= e^{-t^4} \cdot 2t - e^{-t^2}.$$

Inoltre $F'(0) = -1 < 0$, $F(1) = e^{-1} > 0$ e

$$F'(2) = e^{-16} \cdot 4 - e^{-4} = e^{-16}(4 - e^{12}) < 0.$$

Quindi, dato che F' è continua, per il teorema dei valori intermedi $\exists\, t_1 \in (0,1)$ e $\exists\, t_2 \in (1,2)$ tali che $F'(t_1) = F'(t_2) = 0$. $\square$

(b) VERO. Dato che $e^{-x^2} > 0$, il segno di F è

F	$++++$		$----$		$++++$
	$t<t^2$	0	$t^2<t$	1	$t<t^2$

Quindi F ammette un minimo assoluto in $[0,1]$

$$\inf_{\mathbb{R}} F = \inf_{[0,1]} F \underset{\substack{[0,1] \text{ è compatto} \\ \text{e } F \text{ è continua}}}{=} \min_{[0,1]} F < 0.$$

$\square$

(c) FALSO. Per $t \geqslant 0$

$$F(-t) = \int_{-t}^{t^2} e^{-x^2}\,dx = \underbrace{\int_{-t}^{t} e^{-x^2}\,dx}_{\geqslant 0} + \int_{t}^{t^2} e^{-x^2}\,dx \geqslant F(t).$$

Quindi $\sup_{\mathbb{R}} F = \sup_{(-\infty,0)} F$ e non c'è massimo assoluto perché $F'(t) = e^{-2t^4}\cdot 2t - e^{-t^2} < 0$ per $t<0$ e dunque F è strett. decrescente in $(-\infty,0)$. $\square$

(d) VERO. Abbiamo che

$$F''(t) = \left(e^{-t^4}\cdot 2t - e^{-t^2}\right)' = e^{-t^4}(2-8t^4) + 2t\,e^{-t^2}$$

$$= \underbrace{e^{-t^4}}_{>0}\Big(\underbrace{2-8t^4+2t\,e^{t^4-t^2}}_{\to +\infty \text{ per } t\to+\infty}\Big)$$

Quindi $\exists\, a>0$ tale che $F''(t)>0 \ \forall t \geqslant a$ ossia F è strettamente convessa in $[a,+\infty)$. $\square$

(e) VERO. Per il teorema di Taylor (resto Lagrange) per $x<0$ $\exists\, t\in(x,0)$ tale che

$$1+x \leq e^x = 1+x+\frac{e^t x^2}{2} \leqslant 1+x+\frac{x^2}{2}.$$

Quindi per $x\in[0,1]$

$$1-x^2 \leq e^{-x^2} \leqslant 1-x^2+\frac{x^4}{2}.$$

Così

$$\underbrace{2\int_0^1 (1-x^2)\,dx}_{\frac{5}{4} < \frac{4}{3}} \leq F(-1) = 2\int_0^1 e^{-x^2}\,dx \leqslant \underbrace{2\int_0^1\left(1-x^2+\frac{x^4}{2}\right)dx}_{\frac{5}{3} > \frac{23}{15}}. \quad \square$$