Problem 12362
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Proposed by A. Garcia (France).

Evaluate

$$\lim_{n\to\infty}\int_0^{\pi/2}\frac{n}{\big(\sqrt{2}\cos(x)\big)^n+\big(\sqrt{2}\sin(x)\big)^n}\,dx\,.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. After letting $2\sin^2(x)=1-\frac{2t}{n}$, we have

$$\int_0^{\pi/2}\frac{n}{\big(\sqrt{2}\cos(x)\big)^n+\big(\sqrt{2}\sin(x)\big)^n}\,dx=\int_{-n/2}^{n/2}f_n(t)\,dt=2\int_0^{n/2}f_n(t)\,dt$$

where for any $t\in(-n/2,n/2)$,

$$0\leq f_n(t):=\frac{1}{\big(1+\frac{2t}{n}\big)^{n/2}+\big(1-\frac{2t}{n}\big)^{n/2}}\cdot\frac{1}{\sqrt{1-\big(\frac{2t}{n}\big)^2}}.$$

As $n\to+\infty$, we have that

$$0\leq\int_{n/4}^{n/2}f_n(t)\,dt\leq\frac{1}{\big(\frac{3}{2}\big)^{n/2}}\int_0^{n/2}\frac{dt}{\sqrt{1-\big(\frac{2t}{n}\big)^2}}=\frac{n}{2}\left(\frac{2}{3}\right)^{n/2}\int_0^1\frac{ds}{\sqrt{1-s^2}}=\frac{\pi n}{4}\left(\frac{2}{3}\right)^{n/2}\to 0$$

and by the Dominated Convergence Theorem

$$\int_0^{n/4}f_n(t)\,dt\to\int_0^{+\infty}\frac{1}{e^t+e^{-t}}\,dt=\big[\arctan(e^t)\big]_0^{+\infty}=\frac{\pi}{4}$$

because for all $t\geq 0$,

$$I_{[0,n/4]}(t)\cdot f_n(t)\to\frac{1}{e^t+e^{-t}}$$

and for $n\geq 4$,

$$I_{[0,n/4]}(t)\cdot f_n(t)\leq\frac{1}{\big(1+\frac{t}{2}\big)^2}\cdot\frac{2}{\sqrt{3}}.$$

Hence we may conclude that

$$\begin{aligned}\lim_{n\to\infty}\int_0^{\pi/2}\frac{n}{\big(\sqrt{2}\cos(x)\big)^n+\big(\sqrt{2}\sin(x)\big)^n}\,dx&=2\lim_{n\to\infty}\int_0^{n/4}f_n(t)\,dt+2\lim_{n\to\infty}\int_{n/4}^{n/2}f_n(t)\,dt\\&=2\cdot\frac{\pi}{4}+2\cdot 0=\frac{\pi}{2}.\end{aligned}$$

□