Problem 12361
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Proposed by H. Ohtsuka (Japan).

For a non-negative integer k, let $r_{3k} = 0$, $r_{3k+1} = 1$, and $r_{3k+2} = -1$. Prove

$$\sum_{k=0}^{n-1} \binom{2k}{k} = \sum_{k=0}^{n} r_k \binom{2n}{n-k},$$

for every positive integer n.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We show the statement by induction with respect to n.

Base case. The statement holds for $n = 1$:

$$\sum_{k=0}^{1-1} \binom{2k}{k} = 1 = 2r_0 + r_1 = \sum_{k=0}^{1} r_k \binom{2}{1-k}.$$

Inductive step. We assume that statement holds for $n \geq 1$, and we prove that it holds for $n+1$:

$$\begin{aligned}
\sum_{k=0}^{n+1} r_k \binom{2(n+1)}{n+1-k} &= \sum_{k=0}^{n+1} r_k \left(\binom{2n}{n-1-k} + 2\binom{2n}{n-k} + \binom{2n}{n+1-k} \right) \\
&= \sum_{k=0}^{n-1} r_k \binom{2n}{n-(k+1)} + 2\sum_{k=0}^{n} r_k \binom{2n}{n-k} + \sum_{k=0}^{n+1} r_k \binom{2n}{n-(k-1)} \\
&= \sum_{k=1}^{n} r_{k-1} \binom{2n}{n-k} + 2\sum_{k=0}^{n} r_k \binom{2n}{n-k} + \sum_{k=-1}^{n} r_{k+1} \binom{2n}{n-k} \\
&= 2\sum_{k=0}^{n} r_k \binom{2n}{n-k} + \sum_{k=0}^{n} \underbrace{(r_{k-1} + r_{k+1})}_{=-r_k} \binom{2n}{n-k} - \underbrace{r_{-1}}_{=-1} \binom{2n}{n} + \underbrace{r_0}_{=0} \binom{2n}{n+1} \\
&= \sum_{k=0}^{n} r_k \binom{2n}{n-k} + \binom{2n}{n} = \sum_{k=0}^{n-1} \binom{2k}{k} + \binom{2n}{n} \qquad \text{(inductive hypothesis)} \\
&= \sum_{k=0}^{n} \binom{2k}{k}
\end{aligned}$$

where we extended the definition of r_k for any $k \in \mathbb{Z}$, and we noted that $r_{k-1} + r_k + r_{k+1} = 0$. $\square$