Problem 12360
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Evaluate

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{x_{n+1}}-\frac{n^2}{x_n}\right),$$

where $x_n = \sqrt[n]{\sqrt{2!}\sqrt[3]{3!}\cdots\sqrt[n]{n!}}$.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Let $z_n = n\ln(n) - 2n + \frac{\ln(n)^2}{4} + \frac{\ln(2\pi e)\ln(n)}{2}$. Then

$$z_n - z_{n-1} = \ln(n) - 1 + \frac{\ln(n)}{2n} + \frac{\ln(2\pi)}{2n} + \frac{\ln(n)}{4n^2} + O(1/n^2).$$

and

$$\lim_{n\to\infty}\frac{\frac{\ln(n!)}{n}-(z_n-z_{n-1})}{\frac{\ln(n)}{n^2}} = -\frac{1}{4}$$

where we applied the Stirling's approximation

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + \frac{\ln(2\pi)}{2} + O(1/n).$$

Hence, the series $\sum_{k=1}^{\infty}\left(\frac{\ln(k!)}{k}-(z_k-z_{k-1})\right)$ is convergent, which implies that the following limit exists and it is finite,

$$\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\ln(k!)}{k}-z_n\right)=C$$

where C is some real constant.
By the Stolz-Cesaro theorem (0/0 form), we find that

$$\lim_{n\to\infty}\frac{\ln(x_n)-\frac{z_n+C}{n}}{\frac{\ln(n)}{n^2}} = \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\frac{\ln(k!)}{k}-(z_n+C)}{\frac{\ln(n)}{n}} \stackrel{\text{SC}}{=} \lim_{n\to\infty}\frac{\frac{\ln(n!)}{n}-(z_n-z_{n-1})}{\frac{\ln(n)}{n}-\frac{\ln(n-1)}{n-1}} = \frac{1}{4}.$$

Therefore

$$\ln(x_n) = \frac{1}{n}\sum_{k=1}^{n}\frac{\ln(k!)}{k} = \ln(n) - 2 + \frac{\ln(n)^2}{4n} + \frac{\ln(2\pi e)\ln(n)}{2n} + \frac{C}{n} + \frac{\ln(n)}{4n^2}(1+o(1))$$

and

$$x_n = \frac{n}{e^2}\exp\left(\frac{\ln(n)^2}{4n} + \frac{\ln(2\pi e)\ln(n)}{2n} + \frac{C}{n} + \frac{\ln(n)}{4n^2}(1+o(1))\right).$$

Hence, as $n\to+\infty$, we find that

$$\frac{n^2}{x_n} = e^2\left(n - \frac{\ln(n)^2}{4} - \frac{\ln(2\pi e)\ln(n)}{2} - C + O\left(\frac{\ln(n)^4}{n}\right)\right).$$

Finally, by applying the above approximation, it is easy to compute the desired limit

$$\lim_{n\to\infty}\left(\frac{(n+1)^2}{x_{n+1}}-\frac{n^2}{x_n}\right) = e^2.$$

□