Problem 12359
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Proposed by P. Bracken (USA).

Let n be a positive integer. Prove

$$-\frac{\pi+1}{4n}-\frac{1}{8n^2}<\sum_{k=1}^{n}\frac{1}{(2k-1)^2}-2\left(\sum_{k=1}^{n}\frac{(-1)^{k+1}}{2k-1}\right)^2<\frac{\pi-1}{4n}-\frac{1}{8n^2}.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Let

$$S_n=\sum_{k=n}^{\infty}\frac{1}{(2k+1)^2}\qquad\text{and}\qquad T_n=\sum_{k=n}^{\infty}\frac{(-1)^{k-n}}{2k+1}.$$

Then

$$\begin{aligned}\Delta_n=\sum_{k=1}^{n}\frac{1}{(2k-1)^2}-2\left(\sum_{k=1}^{n}\frac{(-1)^{k+1}}{2k-1}\right)^2&=\frac{\pi}{8}-S_n-2\left(\frac{\pi}{4}-(-1)^nT_n\right)^2\\&=-S_n+(-1)^n\pi T_n-2T_n^2.\end{aligned}\tag{1}$$

We investigate the two sums S_n and T_n separately.

Letting $f(x)=1/(2x+1)^2$ in the Euler-Maclaurin summation formula

$$\sum_{k=n}^{\infty}f(k)=\int_n^{\infty}f(x)\,dx+\frac{f(\infty)+f(n)}{2}+\frac{f'(\infty)-f'(n)}{12}+R_n$$

with $|R_n|\le\frac{1}{120}\int_n^{\infty}|f'''(x)|\,dx$, we find

$$S_n=\frac{1}{2(2n+1)}+\frac{1}{2(2n+1)^2}+\frac{1}{3(2n+1)^3}+R_n$$

with $|R_n|\le\frac{1}{5(2n+1)^4}$.

As regards the second sum, we have that

$$T_n=(-1)^n\sum_{k=n}^{\infty}\int_0^1(-x^2)^k\,dx=\int_0^1\frac{x^{2n}}{1+x^2}\,dx=\frac{1}{4n}-\int_0^1\left(\frac{x^{2n}}{2x}-\frac{x^{2n}}{1+x^2}\right)dx=\frac{1}{4n}-E_n$$

where $E_n=\int_0^1\frac{x^{2n-1}(1-x)^2}{2(1+x^2)}\,dx$ and for $n>1$,

$$\frac{1}{8n(2n+1)(n+1)}=\int_0^1\frac{x^{2n-1}(1-x)^2}{4}\,dx\le E_n\le\int_0^1\frac{x^{2n-1}(1-x)^2}{4x^2}\,dx=\frac{1}{8n(2n-1)(n-1)}.$$

Notice that $E_1=(\pi-3)/4$.

Going back to (1), it follows that

$$\begin{aligned}\Delta_n&=-\frac{1}{2(2n+1)}-\frac{1}{2(2n+1)^2}-\frac{1}{3(2n+1)^3}-R_n+(-1)^n\pi\left(\frac{1}{4n}-E_n\right)-2\left(\frac{1}{4n}-E_n\right)^2\\&=-\frac{1}{4n}+\frac{2n+3}{12n(2n+1)^3}-R_n+(-1)^n\pi\underbrace{\left(\frac{1}{4n}-E_n\right)}_{>0}-\frac{1}{8n^2}+\underbrace{\left(\frac{1}{n}-2E_n\right)}_{>0}E_n.\end{aligned}$$

Hence the desired upper bound is proved as soon as

$$\frac{2n+3}{12n(2n+1)^3} - R_n - \Big(\underbrace{\pi - \frac{1}{n} + 2E_n}_{>2}\Big)E_n < 0$$

which holds because the left-hand side is less than

$$\frac{2n+3}{12n(2n+1)^3} + \frac{1}{5(2n+1)^4} - \frac{2}{8n(2n+1)(n+1)} = -\frac{100n^2+108n+23}{60(2n+1)^4(n+1)} < 0.$$

Similarly, the desired lower bound is proved as soon as

$$\frac{2n+3}{12n(2n+1)^3} - R_n + \Big(\underbrace{\pi + \frac{1}{n} - 2E_n}_{>0}\Big)E_n > 0$$

which holds because, the left-hand side is greater than

$$\frac{2n+3}{12n(2n+1)^3} - \frac{1}{5(2n+1)^4} = \frac{20n^2+28n+15}{60(2n+1)^4(n+1)} > 0.$$

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