Problem 12358
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For a positive integer q and a set A of positive integers, say that A is q-good if every sufficiently large integer has exactly q representations as the sum of distinct elements of A.
(a) Which sets A are 1-good?
(b) For which q does there exist a q-good set?
(c) For q as in (b), which sets A are q-good?

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Let A be a q-good set for some $q \geq 1$ and let N be a positive integer such that for any $n > N$ there are exactly q representations as the sum of distinct elements of A. It is clear that A is an infinite set. The first part of the proof, i.e. i), ii) and iii) is a variation of the official solution of problem 3 of APMO-2020.

i) If $a \in A$ with $a > 3N$ and $x \in (a, a+N]$ then $x \notin A$.
Let $y \in (2a - N, 2a)$ then

$$x > a = 2a - a > y - a > y - x > (2a - N) - (a + N) = a - 2N > N.$$

Therefore both $y - x$ and $y - a$ have q representations each in $A \setminus \{a, x\}$. It follows that if $x \in A$ then y, which is $> N$, has $2q \geq q + 1$ representations in A: q representations of $a + (y - a)$ and q representations of $x + (y - x)$. Contradiction.

ii) If $a \in A$ with $a > 3N$ and $x \in (a + N, 2a)$ then $x \notin A$.
Since $x - a \in (N, a)$, it follows that $x - a$ has q representations in $A \setminus \{a\}$. If $x \in A$ then x has $q+1$ representations in A: x itself and q representations of $a + (x - a)$. Contradiction.

iii) If $a \in A$ with $a > 3N$ and then $2a \in A$.
Assume that $2a \notin A$. Since $2a > N$ then $2a$ has q representations in A. None of them contains elements $> a$ by i) and ii). They don't even contain a otherwise $2a - a = a$ has q representations in $A \setminus \{a\}$ plus 1 given by a itself. If each one of these q representations include elements in $[a - N, a)$ then the number of such elements should be > 2 (otherwise their sum is $< a + a = 2a$) and < 3 (otherwise their sum is $> 3(a - N) = 2a + a - 3N > 2a$). So any representation of $2a$ has an element $a' \in [1, a - N)$ and it follows that $2a - a' \in (a + N, 2a)$ has a representation in $A \setminus \{a\}$. On the other hand, $a - a' \in (N, a)$ has q representations in $A \setminus \{a\}$, and therefore $2a - a' = a + (a - a')$ has q representations in A which are different from the previous one because they include a. So $2a - a'$ has at least $q + 1$ representations in A, which is a contradiction.

By i), ii), and iii), we have that there exist a positive integer a, and a finite set $B \subset \{1, 2, \ldots, a-1\}$ such that $A = B \cup \{a, 2a, 2^2a, 2^3a \ldots\}$. The generating function of the number of representation as the sum of distinct elements of A is

$$F(x) = \prod_{b \in B}(1 + x^b) \cdot \prod_{k \geq 0}(1 + x^{2^k a}) = \frac{\prod_{b \in B}(1 + x^b)}{1 - x^a}$$

where we applied the identity

$$\prod_{k \geq 0}(1 + z^{2^k}) = \frac{1}{1 - z}$$

which follows from the uniqueness of the binary representation.
On the other hand, if A is a q-good set then

$$F(x) = \frac{q}{1 - x} + Q(x)$$

where Q is a polynomial. Therefore

$$\prod_{b \in B}(1 + x^b) = q\,\frac{1 - x^a}{1 - x} + (1 - x^a)Q(x)$$

and as $x \to 1^-$ we find that $2^{|B|} = qa$ which implies that q and a are powers of two. In particular, the elements of A are eventually consecutive powers of two.

Finally we show that for all $q = 2^n$ with $n \geq 0$ there is a q-good set A and we give a characterization of such sets: $A = B \cup P$ where $P = \{2^m, 2^{m+1}, 2^{m+2}, \dots\}$ for some $m \geq 0$, and $B = B_1 \cup B_2$ is a finite set such that $B_1 = \{d_0, 2d_1, 2^2d_2, 2^3d_3, \dots, 2^{m-1}d_{m-1}\}$ with $d_0, d_1, \dots, d_{m-1}$ distinct odd integers, and $B_2 \subset \mathbb{N}^+ \setminus (P \cup B_1)$ with $|B_2| = n$. Then the generating function of the number of representation as the sum of distinct elements of A is

$$\begin{aligned} F(x) &= \prod_{b \in B_2} (1 + x^b) \prod_{k=0}^{m-1} (1 + x^{2^k d_k}) \prod_{k \geq m} (1 + x^{2^k}) \\ &= \prod_{b \in B_2} (1 + x^b) \cdot \prod_{k=0}^{m-1} \frac{1 + x^{2^k d_k}}{1 + x^{2^k}} \cdot \frac{1}{1 - x} \\ &= \prod_{b \in B_2} (1 + x^b) \cdot \prod_{k=0}^{m-1} \Big(\sum_{j=0}^{d_k - 1} (-1)^j x^{2^k j} \Big) \cdot \frac{1}{1 - x} \\ &= \frac{R}{1 - x} + Q(x) = \frac{2^n}{1 - x} + Q(x) \end{aligned}$$

where the polynomial division yields a quotient polynomial $Q(x)$ and a constant remainder

$$R = \prod_{b \in B_2} (1 + 1) \prod_{k=0}^{m-1} 1 = 2^{|B_2|} = 2^n.$$

Note that the definition of B_1 follows from the fact that $(1 + x^{2^k})$ divides $(1 + x^b)$ if and only if $b = d \cdot 2^k$ where d is odd. $\square$