Problem 12357
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Proposed by Van Khea (Cambodia) and D. S. Marinescu (Romania).

Suppose that triangles ABC and DEF have the same centroid, where D, E, and F are on the segments BC, CA, and AB, respectively. Let I be the incenter of triangle ABC. Prove

$$\frac{|AI|}{|AD|}+\frac{|BI|}{|BE|}+\frac{|CI|}{|CF|}\leq 2.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Since D, E, and F are on the segments BC, CA, and AB, respectively, then there exist $\alpha,\beta,\gamma\in[0,1]$ such that

$$D=(1-\alpha)B+\alpha C,\quad E=(1-\beta)C+\beta A,\quad F=(1-\gamma)A+\gamma B.$$

Given that the triangles ABC and DEF have the same centroid, it follows that

$$\frac{A+B+C}{3}=\frac{D+E+F}{3}\Leftrightarrow \beta(C-A)+\gamma(A-B)=\alpha(C-B).$$

Therefore, by the law of sines,

$$\beta(B-A)\times(C-A)=\alpha(B-A)\times(C-B)\implies \beta=\frac{ac\sin(\widehat{B})}{cb\sin(\widehat{A})}\cdot\alpha=\alpha$$

where $a=|BC|$, $b=|CA|$, $c=|AB|$. Similarly we find $\gamma=\alpha$.

By the the law of cosines,

$$\begin{aligned}|AD|^2&=|BD|^2+c^2-2|BD|c\cos(\widehat{B})=(\alpha a)^2+c^2+\alpha(b^2-a^2-c^2)\\&=(\alpha b+(1-\alpha)c)^2\frac{s(s-a)}{bc}+(\alpha b-(1-\alpha)c)^2\frac{(s-c)(s-b)}{bc}\\&\geq(\alpha b+(1-\alpha)c)^2\frac{s(s-a)}{bc}\end{aligned}$$

where $s=(a+b+c)/2$. Moreover, it is known that $|AI|^2=\frac{bc(s-a)}{s}$.
Hence, because of the convexity of $x\to 1/x$ for $x>0$, we find

$$\frac{|AI|}{|AD|}\leq\frac{bc/s}{\alpha b+(1-\alpha)c}\leq\frac{\alpha c+(1-\alpha)b}{s}.$$

Analogous inequalities hold for $\frac{|BI|}{|BE|}$ and $\frac{|CI|}{|CF|}$, and, recalling that $\beta=\gamma=\alpha$, we may conclude that

$$\frac{|AI|}{|AD|}+\frac{|BI|}{|BE|}+\frac{|CI|}{|CF|}\leq\frac{1}{s}\left((\alpha c+(1-\alpha)b)+(\alpha a+(1-\alpha)c)+(\alpha b+(1-\alpha)a)\right)=2.$$

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