Problem 12356
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Let $A(z) = z^3 - z^2$ *and* $B(z) = 1 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1}\binom{3n+1}{n} z^{n+1}$. *Prove that* B *is a one-sided inverse to* A *in the sense that* $A(B(z)) = z$. *Also, prove* $B(A(z)) = 1 - z^2 M(-z)$, *where*

$$M(z) = \frac{1 - z - \sqrt{1 - 2z - 3z^2}}{2z^2}.$$

The coefficients of M(z) are the Motzkin numbers $1, 1, 2, 4, 9, 21 \ldots$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We first show that $A(B(z)) = z$, that is

$$B^3(z) - B^2(z) = z \tag{1}$$

in a neighborhood of $z = 0$. Let $f(z, w) = A(w) - z$. Since $f_b(0, 1) = 1 \neq 0$ then, by the Implicit Function Theorem, there is a unique analytic function $B(z)$ such that $f(B(z), z) = 0$ and $B(0) = 1$:

$$B(z) = 1 + \sum_{n=0}^{\infty} a_{n+1} z^{n+1}.$$

Moreover, by the Lagrange Inversion Theorem,

$$\begin{aligned} a_{n+1} &= \frac{1}{(n+1)!} \lim_{w \to 1} \frac{d^n}{dw^n} \left(\left(\frac{w-1}{A(w) - A(1)} \right)^{n+1} \right) = \frac{1}{(n+1)!} \lim_{w \to 1} \frac{d^n}{dw^n} \left(w^{-(2n+2)} \right) \\ &= \frac{(-1)^n (2n+2)(2n+3) \cdots (2n+2+n-1)}{(n+1)!} = \frac{(-1)^n}{n+1} \binom{3n+1}{n} \end{aligned}$$

and we are done.

Finally we show that $B(A(z)) = 1 - z^2 M(-z)$, that is

$$B(z^3 - z^2) = F(z) \tag{2}$$

where

$$F(z) = 1 - z^2 M(-z) = \frac{1 - z + \sqrt{1 + 2z - 3z^2}}{2}.$$

It is straightforward to check that $F^3(z) - F^2(z) = z^3 - z^2$ in a neighborhood of $z = 0$.
Hence, by (1),

$$B^3(z^3 - z^2) - B^2(z^3 - z^2) = z^3 - z^2 = F^3(z) - F^2(z)$$

which implies

$$B^3(z^3 - z^2) - F^3(z) = B^2(z^3 - z^2) - F^2(z),$$

and it follows that

$$(B(z^3 - z^2) - F(z)) \cdot (B^2(z^3 - z^2) + B(z^3 - z^2)F(z) + F^2(z) - B(z^3 - z^2) - F(z)) = 0.$$

Since $B(0) = F(0) = 1$ then the second factor at $z = 0$ is equal to

$$B^2(0) + B(0)F(0) + F^2(0) - B(0) - F(0) = 3 - 2 = 1 \neq 0$$

and, by continuity, it is different from zero in a neighborhood of $z = 0$. Therefore we may conclude that, in such neighborhood, the first factor is identically zero, that is (2) holds. $\square$