Problem 12354
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Let n and k be positive integers with $n \geq 3$, and let

$$p(x) = x^n + x^{n-1} + \cdots + x - k.$$

(a) Prove that the roots of $p(x)$ in the complex plane are simple.
(b) Prove that if $k > n$, then $p(x)$ has at least one root with negative real part and nonzero imaginary part.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. (a) Let x_0 be a multiple root of p then $p(x_0) = p'(x_0) = 0$. Let

$$q(x) = (x-1)p(x) = x^{n+1} - (k+1)x + k$$

then $q(x_0) = 0$ and $q'(x_0) = p(x_0) + (x_0 - 1)p'(x_0) = 0$ and therefore

$$0 = (n+1)q(x_0) - x_0 q'(x_0) = -n(k+1)x_0 + k(n+1) \implies x_0 = \frac{k(n+1)}{n(k+1)} > 0.$$

On the other hand, $p'(x_0) = nx_0^{n-1} + (n-1)x_0^{n-2} + \cdots + 1 > 0$ yielding a contradiction.

(b) Assuming that $k > n \geq 3$, we are going to evaluate the exact number of roots of p with negative real part and nonzero imaginary part. By (a), p has all simple roots. Moreover $p(1) = n - k > 0$ and therefore q has all simple roots too which are the roots of p plus the root $x = 1$. For any real y,

$$q(iy) = \begin{cases} k + i(y^{n+1} - (k+1)y) & \text{if } n \equiv 0 \pmod 4, \\ -y^{n+1} + k - i(k+1)y & \text{if } n \equiv 1 \pmod 4, \\ k + i(-y^{n+1} - (k+1)y) & \text{if } n \equiv 2 \pmod 4, \\ y^{n+1} + k - i(k+1)y & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

It is easy to verify that q has no roots along the imaginary axis and

$$\lim_{y \to \pm\infty} \arg(q(iy)) = \begin{cases} \pm\frac{\pi}{2} & \text{if } n \equiv 0 \pmod 4, \\ \mp\pi & \text{if } n \equiv 1 \pmod 4, \\ \mp\frac{\pi}{2} & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Therefore, by the Routh-Hurwitz theorem,

$$\Delta = L - R = \frac{1}{\pi}\left(\lim_{y\to+\infty} \arg(q(iy)) - \lim_{y\to-\infty} \arg(q(iy))\right) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 4, \\ -2 & \text{if } n \equiv 1 \pmod 4, \\ -1 & \text{if } n \equiv 2 \pmod 4, \\ 0 & \text{if } n \equiv 3 \pmod 4, \end{cases}$$

where L is the number of roots of q in the left half-plane and R the number of roots of q in the right half-plane. Since $L + R = n + 1$, it follows that

$$L = \frac{n+1+\Delta}{2}.$$

Notice that, along the negative real axis, $q > 0$ when n is odd, whereas q has just one root when n is even because $\lim_{x\to-\infty} q(x) = -\infty$, $q(-1) = 2k > 0$, $q(0) = k > 0$ and q' changes sign just once in $(-\infty, 0)$.

The number of roots of p with with negative real part and nonzero imaginary part is the same of q which is equal to

$$L - \begin{cases} 1 & \text{if } n \equiv 0 \pmod 2 \\ 0 & \text{if } n \equiv 1 \pmod 2 \end{cases} = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \pmod 4 \\ \frac{n-1}{2} & \text{if } n \equiv 1 \pmod 4 \\ \frac{n}{2} - 1 & \text{if } n \equiv 2 \pmod 4 \\ \frac{n+1}{2} & \text{if } n \equiv 3 \pmod 4 \end{cases} = 2\left\lfloor \frac{n+1}{4} \right\rfloor$$

which is always greater than 1 for $n \geq 3$ and we are done. □