Problem 12353
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Proposed by Y. Tian (China).

Let A be a square matrix with complex entries, and let A^* denote the conjugate transpose of A. Show that $AA^* = A^*A$ if and only if $\operatorname{rank}(A^2) = \operatorname{rank}(A)$, $A^2A^* = A^*A^2$, *and* $A^3A^* = A^*A^3$.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We first assume that $AA^* = A^*A$. The matrix A is normal and therefore $A = UDU^{-1}$ for some unitary matrix U and a diagonal matrix D whose diagonal entries are the eigenvalues of A (repeated with the corresponding multiplicities). Then $\operatorname{rank}(A)$ is equal to the number of the non-zero diagonal entries of D. Since $A^2 = UD^2U^{-1}$ and the number of the non-zero diagonal entries of D^2 remains the same, it follows that $\operatorname{rank}(A^2) = \operatorname{rank}(A)$. Furthermore, for any $n \geq 2$,

$$A^nA^* = A^{n-1}(AA^*) = A^{n-1}(A^*A) = (A^{n-1}A^*)A = \cdots = A^*A^n.$$

We assume now that $\operatorname{rank}(A^2) = \operatorname{rank}(A)$, $A^2A^* = A^*A^2$, and $A^3A^* = A^*A^3$.
Since $\operatorname{Im}(A^2) \subseteq \operatorname{Im}(A)$, $\operatorname{Ker}(A^2) \supseteq \operatorname{Ker}(A)$, then the assumption $\operatorname{rank}(A^2) = \operatorname{rank}(A)$ and the rank-nullity theorem imply that $\operatorname{Im}(A^2) = \operatorname{Im}(A)$ and $\operatorname{Ker}(A^2) = \operatorname{Ker}(A)$.
Moreover, the matrix A^2 is normal because

$$A^2(A^2)^* = (A^2A^*)A^* = (A^*A^2)A^* = A^*(A^2A^*) = A^*(A^*A^2) = (A^2)^*A^2.$$

Thus the eigenvectors of A^2 corresponding to distinct eigenvalues are necessarily orthogonal which implies

$$\operatorname{Im}(A) = \operatorname{Im}(A^2) \perp \operatorname{Ker}(A^2) = \operatorname{Ker}(A).$$

It follows that there exists a unitary matrix U such that

$$A = U \left[\begin{array}{c|c} M & 0 \\ \hline 0 & 0 \end{array}\right] U^{-1}$$

where M is $r \times r$ invertible matrix with $r = \operatorname{rank}(A)$. Given that $U^{-1} = U^*$, it is easy to verify that $A^2A^* = A^*A^2$ and $A^3A^* = A^*A^3$ imply $M^2M^* = M^*M^2$ and $M^3M^* = M^*M^3$. Hence

$$(MM^*)M^2 = M(M^*M^2) = M(M^2M^*) = M^3M^* = M^*M^3 = (M^*M)M^2$$

and, after multiplying on the right by M^{-2}, we find $MM^* = M^*M$ and finally we get

$$AA^* = U \left[\begin{array}{c|c} MM^* & 0 \\ \hline 0 & 0 \end{array}\right] U^{-1} = U \left[\begin{array}{c|c} M^*M & 0 \\ \hline 0 & 0 \end{array}\right] U^{-1} = A^*A.$$

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