Problem 12351
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Proposed by S. Stewart (Saudi Arabia).

Evaluate

$$\int_0^\infty \frac{\ln(\cos^2(x))\sin^3(x)}{x^3(1+2\cos^2(x))}\,dx.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We have to evaluate $F(1)$ where

$$F(t)=\int_0^\infty \frac{\ln(1-t\sin^2(x))\sin^3(x)}{x^3(3-2\sin^2(x))}\,dx \quad \text{for } t\in[0,1].$$

For $t\in[0,1)$,

$$\begin{aligned}
F'(t)&=\int_0^\infty \frac{-\frac{\sin^5(x)}{x^3}}{(1-t\sin^2(x))(3-2\sin^2(x))}\,dx\\
&=\int_0^\infty \left(\frac{\frac{2/3}{3t-2}}{1-\frac{2}{3}\sin^2(x)}-\frac{\frac{t}{3t-2}}{1-t\sin^2(x)}\right)\frac{\sin^5(x)}{x^3}\,dx\\
&=\frac{1}{3t-2}\sum_{n=0}^\infty\left(\left(\frac{2}{3}\right)^n-t^n\right)\int_0^\infty \frac{\sin^{2n+3}(x)}{x^3}\,dx
\end{aligned}$$

where the interchange of the summation and integration is justified since the series is uniformly convergent. By integrating two times by parts, we find

$$\begin{aligned}
\int_0^\infty \frac{\sin^{2n+3}(x)}{x^3}\,dx&=\frac{(2n+3)}{2}\int_0^\infty \frac{\sin^{2n+2}(x)\cos(x)}{x^2}\,dx\\
&=\frac{(2n+3)(2n+2)}{2}\int_0^\infty \frac{\sin^{2n+1}(x)}{x}\,dx-\frac{(2n+3)^2}{2}\int_0^\infty \frac{\sin^{2n+3}(x)}{x}\,dx\\
&=\frac{(2n+3)(2n+2)}{2}\,\frac{\pi}{2^{2n+1}}\binom{2n}{n}-\frac{(2n+3)^2}{2}\,\frac{\pi}{2^{2n+3}}\binom{2n+2}{n+1}\\
&=\frac{\pi/4}{4^n}\left(1+\frac{1/2}{n+1}\right)\binom{2n}{n}
\end{aligned}$$

where

$$\begin{aligned}
\int_0^\infty \frac{\sin^{2n+1}(x)}{x}\,dx&=\frac{1}{4^n}\sum_{k=0}^n(-1)^k\binom{2n+1}{n-k}\int_0^\infty \frac{\sin((2k+1)x)}{x}\,dx\\
&=\frac{\pi/2}{4^n}\sum_{k=0}^n(-1)^k\binom{2n+1}{n-k}=\frac{\pi}{2^{2n+1}}\binom{2n}{n}
\end{aligned}$$

with $\sin^{2n+1}(x)=\frac{1}{4^n}\sum_{k=0}^n(-1)^k\binom{2n+1}{n-k}\sin((2k+1)x)$ and $\int_0^\infty \frac{\sin(t)}{t}\,dt=\frac{\pi}{2}$.
Hence

$$F'(t)=\frac{\pi}{4}\,\frac{g(2/3)-g(t)}{3t-2}$$

where for $|z|<1$,

$$g(z)=\sum_{n=0}^\infty\binom{2n}{n}\left(\frac{z}{4}\right)^n+\frac{1}{2}\sum_{n=0}^\infty\frac{1}{n+1}\binom{2n}{n}\left(\frac{z}{4}\right)^n=\frac{1}{\sqrt{1-z}}+\frac{1}{1+\sqrt{1-z}}.$$

Finally, since $F(0) = 0$,

$$
\begin{aligned}
F(1) &= F(0) + \int_0^1 F'(t)\,dt = \frac{\pi}{4}\int_0^1 \frac{g(2/3) - g(t)}{3t-2}\,dt \\
&\overset{s=\sqrt{1-t}}{=} -\frac{\pi}{4}\int_0^1 \left(\frac{1}{s+1} + \frac{1}{\sqrt{3}s+1}\right) ds \\
&= -\frac{\pi}{4}\left[\ln(s+1) + \frac{\ln(\sqrt{3}s+1)}{\sqrt{3}}\right]_0^1 \\
&= -\frac{\pi}{4}\left(\ln(2) + \frac{\ln(\sqrt{3}+1)}{\sqrt{3}}\right) \approx -1.0001374365.
\end{aligned}
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