Problem 12350
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What is the smallest positive integer k such that for any quadratic polynomial P with integer coefficients, one of the integers $P(1),\dots,P(k)$ has a zero digit when written in base two?

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Solution. We show that $k_{\min}=7$.

The quadratic polynomial $P(z)=x^2-7x+13$ attains the following values:

$$P(1)=P(6)=(111)_2,\ P(2)=P(5)=(11)_2,\ P(3)=P(4)=(1)_2, P(7)=(1101)_2,$$

and therefore $k_{\min}\geq 7$.
The unique quadratic polynomial which attains specific values at 2, 4, 6 is

$$P(x)=\frac{P(2)}{8}(x-4)(x-6)-\frac{P(4)}{4}(x-2)(x-6)+\frac{P(6)}{8}(x-2)(x-4).$$

We impose that $P(2),P(4),P(6)$ has no zero digit when written in base 2 by letting

$$P(2)=\pm(2^A-1),\ P(4)=2^B-1,\ P(6)=\pm(2^C-1)$$

with $A,B,C\in\mathbb{N}^+$ (without loss of generality we may assume that $P(4)>0$).
We claim that at least one of the values $P(1)$, $P(3)$, $P(5)$, $P(7)$ have a zero digit when written in base 2 and therefore $k_{\min}$ is exactly 7.
We distinguish four cases:

1) If $P(2)>0$ and $P(6)>0$ then $P(3)=\frac{3}{8}2^A+\frac{3}{4}2^B-\frac{1}{8}2^C-1$.
2) If $P(2)<0$ and $P(6)>0$ then $P(5)=\frac{1}{8}2^A+\frac{3}{4}2^B+\frac{3}{8}2^C-\frac{5}{4}$.
3) If $P(2)>0$ and $P(6)<0$ then $P(3)=\frac{3}{8}2^A+\frac{3}{4}2^B+\frac{1}{8}2^C-\frac{5}{4}$.
4) If $P(2)<0$ and $P(6)<0$ then $P(1)=-\frac{15}{8}2^A-\frac{5}{4}2^B-\frac{3}{8}2^C+\frac{7}{2}$.

Now, we examine each case in details.
1.1) If $P(3)=2^D-1$ with $D\in\mathbb{N}^+$ then $3\cdot 2^A+6\cdot 2^B-2^C-8=8(2^D-1)$ that is

$$2^A+2^{A+1}+2^{B+1}+2^{B+2}=2^C+2^{D+3}.$$

Because any integer can be represented in a unique way as the sum of the distinct powers of 2, it follows that $A=B+1$ or $A=B+2$ or $A=B$.
1.1.1) If $A=B+1$ then $2^{A+1}+2^{A+2}=2^C+2^{D+3}$ and

$$\begin{cases}A+1=C\\A+2=D+3\end{cases}\implies A\geq 2 \text{ and } P(7)=7\cdot 2^{A-1}-1=6\cdot 2^{A-1}+2^{A-1}-1=(110\underbrace{1\dots1}_{A-1})_2$$

or

$$\begin{cases}A+1=D+3\\A+2=C\end{cases}\implies A\geq 3 \text{ and } P(7)=29\cdot 2^{A-2}-1=28\cdot 2^{A-2}+2^{A-2}-1=(11100\underbrace{1\dots1}_{A-2})_2.$$

1.1.2) If $A = B + 2$ then $2^{A+2} + 2^{A-1} = 2^C + 2^{D+3}$ and

$$\begin{cases} A - 1 = C \\ A + 2 = D + 3 \end{cases} \implies A \geq 3 \text{ and } P(1) = 7 \cdot 2^{A-2} - 1 = (110\underbrace{1\ldots1}_{A-2})_2$$

or

$$\begin{cases} A - 1 = D + 3 \\ A + 2 = C \end{cases} \implies A \geq 5 \text{ and } P(5) = 25 \cdot 2^{A-4} - 1 = (11000\underbrace{1\ldots1}_{A-4})_2.$$

1.1.3) If $A = B$ then $2^A + 2^{A+3} = 2^C + 2^{D+3}$ and

$$\begin{cases} A = C \\ A + 3 = D + 3 \end{cases} \implies A \geq 4 \text{ and } P \text{ is constant (not quadratic)}$$

or

$$\begin{cases} A = D + 3 \\ A + 3 = C \end{cases} \implies A \geq 4 \text{ and } P(5) = 29 \cdot 2^{A-3} - 1 = (11100\underbrace{1\ldots1}_{A-3})_2.$$

1.2) If $P(3) = -(2^D - 1)$ with $D \in \mathbb{N}^+$ then $3 \cdot 2^A + 6 \cdot 2^B - 2^C - 8 = -8(2^D - 1)$ that is

$$2^{A-1} + 2^A + 2^B + 2^{B+1} + 2^{D+2} = 2^3 + 2^{C-1}.$$

Since the left-hand side is greater than 16, then $C > 4$ and, in a similar way as we did before, we find that the above equation has 4 solutions (A, B, C, D):

$$(3, 2, 6, 2),\ (4, 4, 8, 4),\ (4, 5, 8, 2),\ (5, 3, 8, 4),\ (6, 3, 8, 2).$$

The values of $P(1)$ are $33, 105, 85, 145, 205$ respectively and such values have all a zero digit when written in base 2.

2.1) If $P(5) = 2^D - 1$ with $D \in \mathbb{N}^+$ then

$$2^{A-1} + 2^B + 2^{B+1} + 2^{C-1} + 2^C = 1 + 2^{D+2},$$

which has 2 solutions (A, B, C, D): $(2, 2, 1, 2)$, $(4, 1, 1, 2)$. The values of $P(1)$ are $-9, -29$ respectively and such values have all a zero digit when written in base 2.

2.2) If $P(5) = -(2^D - 1)$ with $D \in \mathbb{N}^+$ then the quadratic polynomial P changes its sign at least 3 times which is impossible.

3.1) If $P(3) = 2^D - 1$ with $D \in \mathbb{N}^+$ then

$$2^{A-1} + 2^A + 2^B + 2^{B+1} + 2^{C-1} = 1 + 2^{D+2},$$

which has 2 solutions (A, B, C, D): $(1, 1, 4, 2)$, $(1, 2, 2, 2)$. The values of $P(7)$ are $-29, -9$ respectively and such values have all a zero digit when written in base 2.

3.2) If $P(3) = -(2^D - 1)$ with $D \in \mathbb{N}^+$ then the quadratic polynomial P changes its sign at least 3 times which is impossible.

4.1) If $P(1) = 2^D - 1$ with $D \in \mathbb{N}^+$ then the quadratic polynomial P changes its sign at least 3 times which is impossible.

4.2) If $P(1) = -(2^D - 1)$ with $D \in \mathbb{N}^+$ then

$$2^{A+2} + 2^{B-1} + 2^{B+1} + 2^{C-2} + 2^{C-1} = 1 + 2^2 + 2^{A-2} + 2^{D+1},$$

which has 2 solutions (A, B, C, D): $(2, 2, 4, 4)$, $(5, 2, 2, 6)$. The values of $P(7)$ are $-33, -21$ respectively and such values have all a zero digit when written in base 2. □