Problem 12349
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Proposed by R. Tauraso (Italy).

Let A_n be the set of permutations of $\{1,\dots,n\}$ that have at least one fixed point. For $\pi \in A_n$ we write Fix(π) for $\{j : \pi(j) = j\}$. Evaluate

$$\sum_{\pi\in A_n}\left(\frac{sgn(\pi)}{|Fix(\pi)|}\sum_{j\in Fix(\pi)} j\right).$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We have that

$$\begin{aligned}
\sum_{\pi\in A_n}\frac{\operatorname{sgn}(\pi)}{|\operatorname{Fix}(\pi)|}\sum_{j\in\operatorname{Fix}(\pi)} j &= \sum_{k=1}^{n}\frac{1}{k}\sum_{\substack{\pi\in A_n\\ \operatorname{Fix}(\pi)=k}}\operatorname{sgn}(\pi)\sum_{j\in\operatorname{Fix}(\pi)} j\\
&=\sum_{k=1}^{n}\frac{1}{k}\cdot\Big(\sum_{\substack{J\subseteq\{1,\dots,n\}\\ |J|=k}}\sum_{j\in J} j\Big)\cdot\sum_{\pi\in D_{n-k}}\operatorname{sgn}(\pi)\\
&=\sum_{k=1}^{n}\frac{1}{k}\cdot\Big(\binom{n-1}{k-1}\sum_{j=1}^{n} j\Big)\cdot\det(M_{n-k})\\
&=\sum_{k=1}^{n}\frac{1}{k}\cdot\frac{n(n+1)}{2}\binom{n-1}{k-1}\cdot(-1)^{n-1-k}(n-1-k)\\
&=(-1)^n\frac{n^2-1}{2}\sum_{k=1}^{n}\binom{n}{k}(-1)^{k+1}-(-1)^n\frac{n(n+1)}{2}\sum_{k=1}^{n}\binom{n-1}{k-1}(-1)^{k-1}\\
&=(-1)^n\frac{n^2-1}{2}(1-(1-1)^n)-(-1)^n\frac{n(n+1)}{2}(1-1)^{n-1}\\
&=(-1)^n\frac{n^2-1}{2}+[n=1]
\end{aligned}$$

where where D_{n-k} is the set of permutations of $\{1,\dots,n-k\}$ with no fixed points, M_{n-k} is the $(n-k)\times(n-k)$ matrix with 0s along the main diagonal and 1s elsewhere, and $[n=1]$ is 1 when $n=1$ and 0 otherwise. $\square$