Problem 12348
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Proposed by E. Vigren and H. Rullgård (Sweden).

We have n people in a circle, numbered from 1 to n clockwise. They are removed one at a time as follows, until just one remains. At each step, remove the nth person among those remaining, where the count starts at the lowest numbered person remaining and procedes clockwise. Let $W(n)$ be the number of the last person remaining.

(a) What is $W(10^{12})$?

(b) For $n \geq 5$, show that $W(n) = n - 4$ if and only if $n/2$ is a Sophie Germain prime (i.e., $n/2$ and $n + 1$ are prime).

(c) Find the smallest even number that does not equal $W(n)$ for any n.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution.
(a) Starting from the final step and working backwards, we verify that $W(n)$ can be computed by the following algorithm written in pseudo-code:

```
function W(n)
    m ← 1
    for k = 2 to n
        r ← n (mod k)
        if (m ≥ r and r > 0) then
            m ← m + 1
        end if
    end for
return m
```

Using C++, after a few minutes, the program returns the value

$$W(10^{12}) = 671046354072.$$

(b) For $n \geq 5$, after the first round, all the odd numbers are removed, in the following order

$$n, 1, 3, \ldots, \begin{cases} n-2 & \text{if } n \equiv 0 \pmod 2, \\ n-1 & \text{if } n \equiv 1 \pmod 2. \end{cases}$$

It follows that $W(n)$ has to be an even number less than n. So if $W(n) = n-4$ then also n is an even number, say $n = 2N$, and the elimination process continues with this list of $N - 1$ even numbers,

$$2, 4, \ldots, n-4, n-2.$$

Moreoever, due to the parity of n, $W(n) = n - 4$ if and only the last two remaining persons are numbered $n - 4$ and $n - 2$ (otherwise $n - 4$ will not be the last to be removed). This happens if and only if

$$\begin{cases} n \not\equiv k & \pmod k \\ n \not\equiv k-1 & \pmod k \end{cases} \qquad \text{for } k = 3, \ldots, N-1$$

that is N and $2N + 1$ have no divisors in the set $\{3, \ldots, N - 1\}$ which means that N and $2N + 1$ are primes, as desired.

(c) We claim that the smallest even number that does not equal $W(n)$ for any n is 34.
The table below shows that all even numbers less than 34 are attained:

$W(n)$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32
n	3	5	7	16	11	13	50	17	19	76	23	56	248	29	31	424

Let $W(n,k)$ be the k-th person to be removed when starting with n people.
It can be checked that $W(n) \neq 34$ for $n \leq 100$. We show that for $n > 100$, the 34-th person is eliminated before the last one, that is $W(n,k) = 34$ for some $k < n$.

If $n \equiv 0 \pmod 2$ then $W(n, \frac{n}{2} + 7) = 34$.

If $n \equiv 3, 9 \pmod{12}$ then $W(n, \frac{2n}{3} + 4) = 34$.

If $n \equiv 5 \pmod{12}$ then $W(n, \frac{5n+11}{6}) = 34$.

If $n \equiv 7, 11 \pmod{12}$ then $W(n, \frac{3n+11}{4}) = 34$.

As regards the last case when $n \equiv 1 \pmod{12}$ we use a different argument.
Since $n - 1$ is divisible by $2, 3, 4, 6, 12$, if $W(n) = 34$ then, among the last 12 persons to be removed, there should be at least 5 persons whose number is less than 34 (see the algorithm given in (a)). Below we show that this is impossible. As noted before, all the 17 odd numbers less than 34 are eliminated after the first round. Moreover the 12 even numbers $2, 4, 6, 8, 10, 14, 16, 20, 24, 26, 28, 32$ are removed at some step k less than $n - 12$ as follows:

$W(n, \frac{n+3}{2}) = 2$, $W(n, \frac{n+5}{2}) = 8$, $W(n, \frac{n+7}{2}) = 14$, $W(n, \frac{n+9}{2}) = 20$, $W(n, \frac{n+11}{2}) = 26$, $W(n, \frac{n+13}{2}) = 32$,

$W(n, \frac{2n+4}{3}) = 4$, $W(n, \frac{2n+7}{3}) = 16$, $W(n, \frac{2n+10}{3}) = 28$, $W(n, \frac{3n+5}{4}) = 6$, $W(n, \frac{3n+9}{4}) = 24$,

$W(n, \frac{4n+6}{5}) = 10$ if $\frac{n-1}{12} \equiv 0 \pmod 5$, $W(n, \frac{5n+7}{6}) = 10$ if $\frac{n-1}{12} \not\equiv 0 \pmod 5$.

Therefore among the last 12 persons there are at most $33 - 17 - 12 = 4$ persons numbered less than 34 and we obtain a contradiction.

□

Remark. The sequence $\{W(n)\}_{n \geq 1}$ appears in OEIS as A128982:

$$1, 1, 2, 2, 4, 2, 6, 2, 6, 6, 10, 2, 12, 2, 6, 8, 16, 2, 18, 2, 16, 18, 22, 2, 22, 12, 16, 8, 28, 2, 30, 2, 28, \ldots$$

It is easy to verify that $W(n) = n - 1$ if and only if n is prime, and $W(n) = 2$ if and only if $n - 1$ is prime.