Problem 12347
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Proposed by M. Tetiva (Romania).

Let a and b be real numbers with $0 < a < 1 < b$. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(0) = 0$ and $f(f(x)) - (a+b)f(x) + abx = 0$ for all $x \in \mathbb{R}$.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We will prove that there are only four solutions:

$$f(x) = ax, \quad f(x) = bx, \quad f(x) = \begin{cases} ax & \text{if } x \geq 0 \\ bx & \text{if } x < 0 \end{cases}, \quad f(x) = \begin{cases} bx & \text{if } x \geq 0 \\ ax & \text{if } x < 0 \end{cases}.$$

It is easy to verify that they satisfy all the assumptions. It remains to show that there are no other solutions.

The function f is injective: $f(x) = f(y)$ implies

$$0 = f(f(x)) - f(f(y)) - (a+b)(f(x) - f(y)) + ab(x-y) = ab(x-y) \implies x = y.$$

Since $f \in C(\mathbb{R})$, it follows that f is strictly monotone. It can't be strictly decreasing, otherwise $f(1) < f(0) = 0$, $f(f(1)) > f(0) = 0$ which yields a contradiction,

$$0 = f(f(1)) + (a+b)(-f(1)) + ab > 0.$$

Therefore f is strictly increasing and bijective with inverse f^{-1}. Notice that from the given identity, we easily find that $f(\mathbb{R}) = \mathbb{R}$.

The function f has a unique fixed point which is $x = 0$:

$$f(x) = x \implies 0 = f(f(x)) - (a+b)f(x) + abx = x - (a+b)x + abx = x(1-a)(1-b) \implies x = 0.$$

Thus, by continuity, we have two cases: either $f(x) > x$ for all $x > 0$, or $f(x) < x$ for all $x > 0$ (a similar discussion can be done for $x < 0$).

Because of the given identity, if $x \in \mathbb{R}$ then the iterates $\{f^{(n)}(x)\}_{n \in \mathbb{Z}}$ satisfy the homogeneous linear recurrence of second order,

$$f^{(n)}(x) - (a+b)f^{(n-1)}(x) + abf^{(n-2)}(x) = 0.$$

Its characteristic polynomial is $z^2 - (a+b)z + ab = (z-a)(z-b)$ and the roots a, b are distinct, therefore

$$f^{(n)}(x) = c_1(x)a^n + c_2(x)b^n = \frac{bx - f(x)}{b-a}\, a^n + \frac{f(x) - ax}{b-a}\, b^n. \tag{1}$$

If $f(x) < x$ for all $x > 0$ then $f^{(n)}(x)$ is a positive decreasing sequence which converges to the unique fixed point $x = 0$. Since $0 < a < 1 < b$, taking the limit as $n \to +\infty$ in (1), we obtain the value 0 if and only if $c_2(x) = 0$, that is $f(x) = ax$.
If $f(x) > x$ for all $x > 0$ then $f^{(-n)}(x)$ is a positive decreasing sequence which converges to the unique fixed point $x = 0$. Since $0 < a < 1 < b$, taking the limit as $n \to -\infty$ in (1), we obtain the value 0 if and only if $c_1(x) = 0$, that is $f(x) = bx$. □