Problem 12346
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Prove that there are infinitely many integers A such that, for every nonzero integer x and distinct positive odd integers m and n, the integer $x^m + Ax^n$ is not a perfect square.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We claim that we can take $A = p^2$ where p is a prime of the form $16k + 7$ with $k \geq 0$. This is an infinite family of numbers because of the Dirichlet's theorem on arithmetic progressions.

Let m and n be distinct positive odd integers and let x be a nonzero integer. We assume that

$$x^m + p^2x^n = z^2$$

holds for some integer z and show that this leads to a contradiction.
We have that $x > 0$ and $z \neq 0$. We distinguish two cases, i) $n > m$ and ii) $m > n$.

i) If $n > m$ then

$$x^m + p^2x^n = x^m(1 + p^2x^{2j}) = z^2 \tag{1}$$

where $2j = n - m$ with $j \geq 1$. Since $\gcd(x, 1 + p^2x^{2j}) = 1$ and m is odd then x is a perfect square, say y^2. Let $w = z/y^m$ then by (1),

$$w^2 - (px^j)^2 = 1$$

which holds if and only $|w| = 1$ and $px^j = 0$ against the fact that $x \neq 0$.

ii) If $m > n$ then

$$x^m + p^2x^n = x^n(x^{2j} + p^2) = z^2 \tag{2}$$

where $2j = m - n$ with $j \geq 1$. Let q be a prime divisor of x.
If $q \neq p$ and $q^a \mid x$ but $q^{a+1} \nmid x$ then $\gcd(q, x^{2j} + p^2) = 1$, and, since n is odd, it follows that a is even. If $q = p$ and $p^a \mid x$ but $p^{a+1} \nmid x$ then $x = p^ay$ and

$$x^n(x^{2j} + p^2) = (p^ay)^n((p^ay)^{2j} + p^2) = p^{na+2}y^n(p^{2(aj-1)}y^{2j} + 1) = z^2.$$

If $aj > 1$ then $\gcd((p^{2(aj-1)}y^{2j} + 1, p) = 1$ and therefore $na + 2$ is even which implies that a is even. Notice that the case $aj = 1$ is impossible, otherwise $a = j = 1$ getting

$$p^{n+2}y^n(y^2 + 1) = z^2.$$

Since $\gcd(p^{n+2}(y^2 + 1), y) = 1$ and n is odd, then y is a perfect square, say b^2, and $y^2 + 1 = pc^2$ for some nonzero integer c. Hence we have that

$$b^4 + 1 = pc^2$$

and, considering the possible values modulo 16, it is easy to verify that the left-hand side is congruent to $0 + 1 \equiv 1$ or $1 + 1 \equiv 2$, whereas the right-hand side is congruent to $7 \cdot 0 \equiv 0$, $7 \cdot 1 \equiv 7$, $7 \cdot 4 \equiv 12$, or $7 \cdot 9 \equiv 15$ getting a contradiction.
Thus, in any case, if q be a prime divisor of x and $q^a \mid x$ but $q^{a+1} \nmid x$ then a is even which implies that x is a perfect square, say y^2. Let $w = z/y^n$ then by (2),

$$p^2 = w^2 - y^{4j} = (w - y^{2j})(w + y^{2j}) \implies \begin{cases} w - y^{2j} = 1 \\ w + y^{2j} = p^2 \end{cases} \implies \frac{p^2 - 1}{2} = y^{2j}.$$

On the other hand,

$$\frac{p^2 - 1}{2} = \frac{(p - 1)(p + 1)}{2} = \frac{(16k + 6)(16k + 8)}{2} = 2^3(8k + 3)(2k + 1)$$

which can not be a perfect square because of the factor 2^3 (the other factors are odd). □