Problem 12344
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Evaluate

$$\int_{-1}^{1} \frac{\arccos(x)}{x^2+x+1}\,dx.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We first note that applying the half-angle formula $\tan(t/2) = \frac{\sin(t)}{1+\cos(t)}$, we find that

$$\arccos(x) = 2\arctan\left(\frac{\sqrt{1-x^2}}{1+x}\right) \qquad \text{for } x \in (-1,1].$$

Therefore, after letting $t = \frac{\sqrt{1-x^2}}{1+x}$, then $x = \frac{1-t^2}{1+t^2}$ and

$$\int_{-1}^{1} \frac{\arccos(x)}{x^2+x+1}\,dx = \int_0^{\infty} \frac{8t\arctan(t)}{t^4+3}\,dt = 4I(1)$$

where

$$I(a) = \int_{-\infty}^{\infty} \frac{t\arctan(at)}{t^4+3}\,dt.$$

By the residue theorem, for $a>0$, we have that

$$\begin{aligned}
I'(a) &= \int_{-\infty}^{\infty} \frac{t^2}{(1+a^2t^2)(t^4+3)}\,dt = 2\pi i\left(\mathrm{Res}(f,i/a) + \mathrm{Res}(f,z_+) + \mathrm{Res}(f,z_-)\right)\\
&= 2\pi i\left(\frac{t^2}{(2a^2t)(t^4+3)}\right)_{t=i/a} + 2\pi i\left(\frac{t^2}{4t^3(1+a^2t^2)}\right)_{t=z_+} + 2\pi i\left(\frac{t^2}{4t^3(1+a^2t^2)}\right)_{t=z_-}\\
&= -\frac{\pi a}{1+3a^4} + \frac{\pi i}{2z_+(1+a^2\sqrt{3}i)} + \frac{\pi i}{2z_-(1-a^2\sqrt{3}i)}\\
&= -\frac{\pi a}{1+3a^4} + \frac{\pi}{\sqrt{2}\sqrt[4]{3}}\cdot\frac{1+\sqrt{3}a^2}{1+3a^4}
\end{aligned}$$

where $z_\pm = \frac{\sqrt[4]{3}}{\sqrt{2}}(\pm 1 + i)$ are the two solutions of the equation $t^4+3=0$ contained in the upper-half plane in $\mathbb{C}$. Hence,

$$\begin{aligned}
\int_{-1}^{1} \frac{\arccos(x)}{x^2+x+1}\,dx &= 4I(1) = 4\left(I(0) + \int_0^1 I'(a)\,da\right)\\
&= 0 - 4\pi\int_0^1 \frac{a}{1+3a^4}\,da + \frac{4\pi}{\sqrt{2}\sqrt[4]{3}}\int_0^1 \frac{1+\sqrt{3}a^2}{1+3a^4}\,da\\
&= -4\pi\left[\frac{\arctan(\sqrt{3}a^2)}{2\sqrt{3}}\right]_0^1 + \frac{4\pi}{\sqrt{2}\sqrt[4]{3}}\left[\frac{\arctan(\sqrt{2}\sqrt[4]{3}a+1) + \arctan(\sqrt{2}\sqrt[4]{3}a-1)}{\sqrt{2}\sqrt[4]{3}}\right]_0^1\\
&= \frac{2\pi}{\sqrt{3}}\left(-\frac{\pi}{3} + \arctan(\sqrt{2}\sqrt[4]{3}+1) + \arctan(\sqrt{2}\sqrt[4]{3}-1)\right)\\
&= \frac{2\pi}{\sqrt{3}}\left(\frac{2\pi}{3} - \arctan\left(\frac{\sqrt[4]{3}(\sqrt{3}+1)}{\sqrt{2}}\right)\right) = \frac{2\pi}{\sqrt{3}}\left(\frac{2\pi}{3} - \arctan\left(\sqrt{3+2\sqrt{3}}\right)\right)
\end{aligned}$$

where at the last line we applied the identity

$$\arctan(x) + \arctan(y) = \pi + \arctan\left(\frac{x+y}{1-xy}\right) \qquad \text{for } x,y>0 \text{ and } xy>1.$$

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