**Problem 12343**
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Proposed by Tran Quang Hung (Vietnam).

*Let $ABCD$ be a convex quadrilateral with $AB = a$, $BC = b$, $CD = c$, $DA = d$, $AC = e$, and $BD = f$. Prove that $ABCD$ is a cyclic quadrilateral if and only if*

$$\frac{f^2 - e^2}{ac + bd} = \frac{(a^2 - c^2)(b^2 - d^2)}{(ab + cd)(ad + bc)}.$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

*Solution.* Applying the law of cosines to the triangles $ABD$ and $BCD$, we have

$$f^2 = a^2 + d^2 - 2ad\cos(A), \qquad f^2 = b^2 + c^2 - 2bc\cos(C),$$

which imply

$$f^2 = \frac{(ab + cd)(ac + bd) - 2abcd(\cos(A) + \cos(C))}{ad + bc}.$$

In the same way we get

$$e^2 = \frac{(ac + bd)(ad + bc) - 2abcd(\cos(B) + \cos(D))}{ab + cd}.$$

Hence

$$\frac{f^2 - e^2}{ac + bd} - \frac{(a^2 - c^2)(b^2 - d^2)}{(ab + cd)(ad + bc)} = \frac{2abcd}{ac + bd} \cdot \left(\frac{\cos(B) + \cos(D)}{ab + cd} - \frac{\cos(A) + \cos(C)}{ad + bc}\right)$$

and therefore the given equality holds if and only if

$$\frac{\cos(A) + \cos(C)}{ad + bc} = \frac{\cos(B) + \cos(D)}{ab + cd}. \qquad (1)$$

The convex quadrilateral is cyclic if and only if $A + C = \pi$.

If $A + C = \pi$ then $B + D = 2\pi - A - C = \pi$,

$$\cos(A) + \cos(C) = \cos(A) + \cos(\pi - A) = 0, \qquad \cos(B) + \cos(D) = \cos(B) + \cos(\pi - B) = 0$$

and therefore the identity (1) holds.

Now we assume that $A + C \neq \pi$. We consider the case when $A + C > \pi$ (the other case can be treated in a similar way). Then $B + D = 2\pi - A - C < \pi$,

$$\cos(A) + \cos(C) < \cos(A) + \cos(\pi - A) = 0, \qquad \cos(B) + \cos(D) > \cos(B) + \cos(\pi - B) = 0$$

and therefore the identity (1) does not hold because the left-hand side is negative and the right-hand side is positive. □