Problem 12341
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Proposed by G. Apostolopoulos (Greece).

Let $x_1, \ldots, x_n$ *be positive real numbers with* $\sum_{i=1}^n x_i^2 \leq n$. *Prove*

$$\prod_{i=1}^n \left(1 + \frac{1}{x_i x_{i+1}}\right)^{x_i^2} \geq 2^{\frac{1}{n}(\sum_{i=1}^n x_i)^2}$$

where x_{n+1} *is taken to be* x_1.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We will show a more general inequality: let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be positive real numbers then

$$\prod_{i=1}^n \left(1 + \frac{1}{y_i}\right)^{x_i^2} \geq \left(1 + \frac{1}{\frac{1}{n}\sum_{i=1}^n y_i}\right)^{\frac{1}{n}(\sum_{i=1}^n x_i)^2}. \tag{1}$$

The required inequality follows from (1) by letting $y_i = x_i x_{i+1}$ and noting that by the rearrangement inequality,

$$\sum_{i=1}^n y_i = \sum_{i=1}^n x_i x_{i+1} \leq \sum_{i=1}^n x_i^2 \leq n \implies 1 + \frac{1}{\frac{1}{n}\sum_{i=1}^n y_i} \geq 2.$$

Now we prove (1) after taking the logarithm of both sides:

$$\begin{aligned}
\log\left(\prod_{i=1}^n \left(1 + \frac{1}{y_i}\right)^{x_i^2}\right) &= \sum_{i=1}^n x_i^2 \log\left(1 + \frac{1}{y_i}\right) \\
&= \sum_{i=1}^n x_i^2 \int_0^1 \frac{dt}{y_i + t} \\
&= \int_0^1 \sum_{i=1}^n \frac{x_i^2}{y_i + t}\, dt \\
&\geq \int_0^1 \frac{\left(\sum_{i=1}^n x_i\right)^2}{\sum_{i=1}^n (y_i + t)}\, dt \qquad \text{(Cauchy-Schwarz inequality)} \\
&= \frac{1}{n}\Big(\sum_{i=1}^n x_i\Big)^2 \int_0^1 \frac{dt}{\frac{1}{n}\sum_{i=1}^n y_i + t} \\
&= \frac{1}{n}\Big(\sum_{i=1}^n x_i\Big)^2 \log\left(1 + \frac{1}{\frac{1}{n}\sum_{i=1}^n y_i}\right).
\end{aligned}$$

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