Problem 12340
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Proposed by A. Garcia (France).

Let $g : [0,1] \to \mathbb{R}$ *be continuous. Prove that*

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n}\, dx = Cg(1/2)$$

for some constant C *(independent of* g*) and determine the value of* C.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Let $x = \frac{1}{2} + \frac{t}{2n}$, then

$$\frac{n}{2^n} \int_0^1 \frac{dx}{x^n + (1-x)^n} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{I_{[-n,n]}(t)}{\left(1+\frac{t}{n}\right)^n + \left(1-\frac{t}{n}\right)^n}\, dt \to \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{e^u + e^{-u}}\, du = \frac{\pi}{4}$$

where for any $t \in \mathbb{R}$, and for any integer $n \geq 2$,

$$0 \leq f_n(t) := \frac{I_{[-n,n]}(t)}{\left(1+\frac{t}{n}\right)^n + \left(1-\frac{t}{n}\right)^n} \leq \frac{1}{\left(1+\frac{|t|}{2}\right)^2}, \quad \text{and} \quad f_n(t) \to \frac{1}{e^t + e^{-t}}.$$

Hence, by the Dominated Convergence Theorem,

$$\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{1}{x^n + (1-x)^n}\, dx = \frac{1}{2} \int_{-\infty}^{+\infty} \frac{1}{e^t + e^{-t}}\, dt = \frac{1}{2} \left[\arctan(e^t)\right]_{-\infty}^{+\infty} = \frac{\pi}{4}.$$

Finally,

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x)}{x^n + (1-x)^n}\, dx &= g(1/2) \cdot \lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{dx}{x^n + (1-x)^n} + \lim_{n\to\infty} \frac{n}{2^n} \int_0^1 \frac{g(x) - g(1/2)}{x^n + (1-x)^n}\, dx \\ &= \frac{\pi}{4} \cdot g(1/2)\end{aligned}$$

as soon as we show that the second limit is zero.
Indeed, by the continuity of g, $\sup_{x\in[0,1]} |g(x)| = M$ is finite, and for any $\varepsilon > 0$ there is $0 < \delta < 1/2$ such that $|g(x) - g(1/2)| < \varepsilon$ for all $x \in [1/2 - \delta, 1/2 + \delta]$. Therefore, noting that $x \to x^n + (1-x)^n$ is decreasing in $[0, 1/2]$ and it is symmetric with respect to $x = 1/2$, we find

$$\begin{aligned}\left|\frac{n}{2^n} \int_0^1 \frac{g(x) - g(1/2)}{x^n + (1-x)^n}\, dx\right| &\leq \varepsilon \cdot \frac{n}{2^n} \int_{1/2-\delta}^{1/2+\delta} \frac{dx}{x^n + (1-x)^n} + 2M \cdot \frac{n}{2^n} \int_{[0,1/2-\delta]\cup[1/2+\delta,1]} \frac{dx}{x^n + (1-x)^n} \\ &\leq \varepsilon \cdot \frac{n}{2^n} \int_0^1 \frac{dx}{x^n + (1-x)^n} + 2M \cdot \frac{n}{2^n} \cdot \frac{(1-2\delta)}{(1/2-\delta)^n + (1/2+\delta)^n} \\ &\leq \varepsilon \cdot M' + 2M \cdot \frac{n(1-2\delta)}{(1+2\delta)^n}\end{aligned}$$

where M' is an upperbound of the converging sequence $\frac{n}{2^n} \int_0^1 \frac{dx}{x^n + (1-x)^n}$. Since $n/(1+2\delta)^n \to 0$ and ε is an arbitrary positive number, we are done. $\square$