Problem 12338
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Proposed by I. Mező (China).

Prove

$$\int_0^\infty \frac{\cos(x)-1}{x(e^x-1)}\,dx = \frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right).$$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We have that

$$\begin{aligned}
\int_0^\infty \frac{\cos(x)-1}{x(e^x-1)}\,dx &= \int_0^\infty \frac{\cos(x)-1}{x}\cdot\frac{e^{-x}}{1-e^{-x}}\,dx \\
&= \int_0^\infty \frac{\cos(x)-1}{x}\cdot\sum_{n=1}^\infty e^{-nx}\,dx \\
&= \sum_{n=1}^\infty \int_0^\infty \frac{\cos(x)-1}{x}\cdot e^{-nx}\,dx \quad \text{(Fubini-Tonelli theorem)} \\
&= \sum_{n=1}^\infty \mathcal{L}\left(\frac{\cos(x)-1}{x}\right)(n) \quad \text{(Laplace transform)} \\
&= \sum_{n=1}^\infty \int_n^\infty \mathcal{L}\left(\cos(x)-1\right)(s)\,ds \quad \text{(Frequency-domain integration)} \\
&= \sum_{n=1}^\infty \int_n^\infty \left(\frac{s}{s^2+1}-\frac{1}{s}\right)ds = \sum_{n=1}^\infty \left[\frac{1}{2}\ln\left(\frac{s^2+1}{s^2}\right)\right]_n^\infty \\
&= -\frac{1}{2}\sum_{n=1}^\infty \ln\left(1+\frac{1}{n^2}\right) = -\frac{1}{2}\ln\left(\prod_{n=1}^\infty\left(1+\frac{1}{n^2}\right)\right) \\
&= -\frac{1}{2}\ln\left(\frac{\sin(i\pi)}{i\pi}\right) \quad \text{(Euler infinite product)} \\
&= \frac{1}{2}\ln\left(\frac{\pi}{\sinh(\pi)}\right)
\end{aligned}$$

where at beginning we applied Fubini-Tonelli theorem,

$$\begin{aligned}
\int_0^\infty \sum_{n=1}^\infty \left|\frac{\cos(x)-1}{x}\cdot e^{-nx}\right| dx &= \int_0^\infty \frac{1-\cos(x)}{x^2}\cdot\sum_{n=1}^\infty xe^{-nx}\,dx \\
&\leq \frac{1}{2}\sum_{n=1}^\infty \int_0^\infty xe^{-nx}\,dx = \frac{1}{2}\sum_{n=1}^\infty \frac{1}{n^2} < +\infty,
\end{aligned}$$

and at the end we used the Euler infinite product for the sine function,

$$\frac{\sin(z)}{z} = \prod_{n=1}^\infty \left(1-\frac{z^2}{n^2\pi^2}\right).$$

□