Problem 12337
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For $k \in \{0, 1, 2\}$, *let*

$$S_k = \sum \frac{(-4)^n}{2n+1} \binom{2n}{n}^{-1},$$

where the sum is taken over all nonnegative integers n *that are congruent to* k *modulo* 3. *Prove*

(a) $\quad S_0 = \dfrac{\ln(1+\sqrt{2})}{3\sqrt{2}} + \dfrac{\pi}{6},$

(b) $\quad S_1 = \dfrac{\ln(1+\sqrt{2})}{3\sqrt{2}} - \dfrac{\ln(2+\sqrt{3})}{2\sqrt{3}} - \dfrac{\pi}{12},$

(c) $\quad S_2 = \dfrac{\ln(1+\sqrt{2})}{3\sqrt{2}} + \dfrac{\ln(2+\sqrt{3})}{2\sqrt{3}} - \dfrac{\pi}{12}.$

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We first note that

$$I_0 = \int_0^1 \frac{ds}{2-s^2} = \frac{1}{2\sqrt{2}} \int_0^1 \left(\frac{1}{\sqrt{2}+s} + \frac{1}{\sqrt{2}-s} \right) ds = \frac{1}{2\sqrt{2}} \left[\ln\left(\frac{\sqrt{2}+s}{\sqrt{2}-s} \right) \right]_0^1 = \frac{\ln(1+\sqrt{2})}{\sqrt{2}},$$

$$\begin{aligned} I_1 &= \int_0^1 \frac{s^2-1}{s^4-s^2+1}\, ds = -\frac{1}{2\sqrt{3}} \int_0^1 \left(\frac{2s+\sqrt{3}}{s^2+\sqrt{3}s+1} - \frac{2s-\sqrt{3}}{s^2-\sqrt{3}s+1} \right) \\ &= -\frac{1}{2\sqrt{3}} \left[\ln\left(\frac{s^2+\sqrt{3}s+1}{s^2-\sqrt{3}s+1} \right) \right]_0^1 = -\frac{\ln(2+\sqrt{3})}{\sqrt{3}}, \\ I_2 &= \int_0^1 \frac{s^2+1}{s^4-s^2+1}\, ds = \frac{1}{2} \int_0^1 \left(\frac{1}{s^2+\sqrt{3}s+1} + \frac{1}{s^2-\sqrt{3}s+1} \right) \\ &= \left[\arctan\left(2s+\sqrt{3}\right) + \arctan\left(2s-\sqrt{3}\right) \right]_0^1 = \frac{\pi}{2}. \end{aligned}$$

From Beta function properties we have

$$\frac{(-4)^n}{2n+1} \binom{2n}{n}^{-1} = (-4)^n \int_0^1 t^n (1-t)^n\, dt = \frac{1}{2} \int_{-1}^1 (s^2-1)^n\, ds = \int_0^1 (s^2-1)^n\, ds$$

with $s = 2t-1$. It follows that

$$\begin{aligned} S_0 + S_1 + S_2 &= \sum_{n=0}^{\infty} \frac{(-4)^n}{2n+1} \binom{2n}{n}^{-1} = \int_0^1 \sum_{n=0}^{\infty} (s^2-1)^n\, ds = \int_0^1 \frac{ds}{1-(s^2-1)} = I_0, \\ S_0 &= \int_0^1 \sum_{n=0}^{\infty} (s^2-1)^{3n}\, ds = \int_0^1 \frac{ds}{1-(s^2-1)^3} = \frac{1}{3} \int_0^1 \left(\frac{1}{2-s^2} + \frac{s^2+1}{s^4-s^2+1} \right) ds = \frac{I_0+I_2}{3}, \\ S_1 - S_2 &= \int_0^1 \sum_{n=0}^{\infty} \left((s^2-1)^{3n+1} - (s^2-1)^{3n+2} \right) ds = \int_0^1 \frac{(s^2-1)(2-s^2)}{1-(s^2-1)^3}\, ds = \int_0^1 \frac{s^2-1}{s^4-s^2+1}\, ds = I_1, \end{aligned}$$

and, after solving the linear system, we find

$$\begin{aligned} S_0 &= \frac{I_0+I_2}{3} = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} + \frac{\pi}{6}, \\ S_1 &= \frac{2I_0+3I_1-I_2}{6} = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} - \frac{\ln(2+\sqrt{3})}{2\sqrt{3}} - \frac{\pi}{12}, \\ S_2 &= \frac{2I_0-3I_1-I_2}{6} = \frac{\ln(1+\sqrt{2})}{3\sqrt{2}} + \frac{\ln(2+\sqrt{3})}{2\sqrt{3}} - \frac{\pi}{12}. \end{aligned}$$

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