Problem 12335
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A Gaussian integer is a complex number $z = a + ib$ for integers a and b. Show that every Gaussian integer can be written in at most one way as a sum of distinct powers of $1+i$, and that the Gaussian integer z can be expressed as such a sum if and only if $i - z$ cannot.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. Let $\mathbb{Z}[i] = \{z = a + ib : a, b \in \mathbb{Z}\}$ be the set of Gaussian integers. We introduce the function $T : \mathbb{Z}[i] \to \mathbb{Z}[i]$,

$$T(z) = \frac{z - (N(z))_2}{1+i} = \frac{(a + ib - r)(1-i)}{2} = \frac{a+b-r}{2} + i\,\frac{b-a+r}{2} \tag{1}$$

where $N(z) = a^2 + b^2$ is the norm of z, and $r = (N(z))_2 = (a+b)_2 \in \{0,1\}$ is the remainder of the division of $N(z)$ by 2.
Now we show that for any $z \in \mathbb{Z}[i]$ the sequence of iterates $\{T^n(z)\}_{n\geq 0}$ is eventually constant and such constant, say $L(z)$, can be 0 or i. Indeed, by (1),

$$N(T(z)) = \frac{(a+b-r)^2 + (b-a+r)^2}{4} = \frac{(a-r)^2 + b^2}{2} < N(z) = a^2 + b^2$$

if and only if $a^2 + b^2 > 0$ and $r = 0$, or $(a+1)^2 + b^2 > 2$ and $r = 1$, that is

$$z \notin S = \{0, i, -i, -1, -2+i, -2-i\}.$$

Hence $N(T^n(z))$ is strictly decreasing until $T^n(z) \in S$, thereafter

$$\begin{aligned}
&-2-i \xrightarrow{T} -2+i \xrightarrow{T} -1+2i \xrightarrow{T} 2i \xrightarrow{T} 1+i \xrightarrow{T} 1 \xrightarrow{T} 0 \circlearrowright\\
&-i \xrightarrow{T} -1 \xrightarrow{T} -1+i \xrightarrow{T} i \circlearrowright
\end{aligned}$$

and we are done.
Moreover $L(z) = 0$ if and only if $L(i-z) = i$: if $(N(z))_2 = r$ then $(N(i-z))_2 = 1 - r$ and by (1),

$$\begin{aligned}
T(i-z) = T(-a + i(1-b)) &= \frac{-a + (1-b) - (1-r)}{2} + i\,\frac{(1-b) + a + (1-r)}{2}\\
&= -\frac{a+b-r}{2} + i\left(1 - \frac{b-a+r}{2}\right) = i - T(z).
\end{aligned}$$

Hence, for all $n \geq 0$,

$$T^n(i-z) = T^{n-1}(i - T(z)) = T^{n-2}(i - T(T(z))) = \cdots = i - T^n(z)$$

which implies that $L(i-z) = i - L(z)$.
So if $L(z) = 0$ then $T^n(z) = 0$ for some positive integer n and

$$\begin{aligned}
z &= T(z)(1+i) + r_0 = T(T(z))(1+i)^2 + r_1(1+i) + r_0\\
&= \cdots = T^n(z)(1+i)^n + \sum_{k=0}^{n-1} r_k(1+i)^k = \sum_{k=0}^{n-1} r_k(1+i)^k
\end{aligned}$$

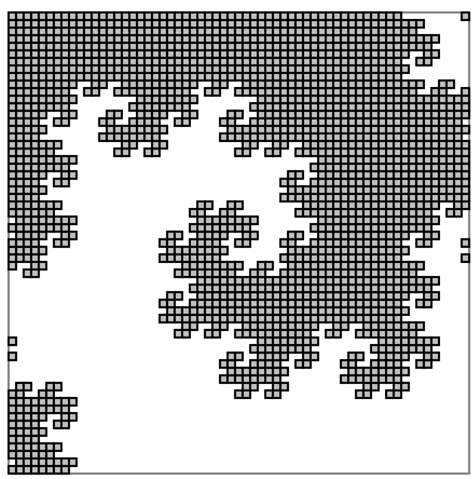
that is $z \in \mathbb{Z}[i]$ can be represented as a sum of distinct powers of $1+i$.

On the other hand, if z can be represented as a sum of distinct powers of $1+i$, that is $z = \sum_{k=0}^{n-1} a_k(1+i)^k$ with $a_0,\ldots,a_{n-1} \in \{0,1\}$ then, due to the fact that $(1+i)^k = 2i(1+i)^{k-2}$ for $k \geq 2$,

$$z \equiv a_1(1+i) + a_0 \pmod 2 \implies N(z) \equiv (a_0+a_1)^2 + a_1^2 \equiv a_0 \pmod 2 \implies r_0 = (N(z))_2 = a_0.$$

By applying the same argument, we obtain $T^n(z) = 0$, $L(z) = 0$, and $r_k = a_k$ for $k = 0,\ldots,n-1$. Hence, we may conclude that $z \in \mathbb{Z}[i]$ can be written as a sum of distinct powers of $1+i$ if and only if $L(z) = 0$. Moreover, it follows also that such representation is unique. Since $L(z) = 0$ if and only if $L(i-z) = i$, then z has such representation if and only if $i - z$ has not. $\square$

Remark. As noted by William Gilbert in *Fractal geometry derived from complex bases*, *The Mathematical Intelligencer*, June 1982, if we draw a grid in the complex plane where each unit square is centered at a Gaussian integer and we shade all the squares whose center can be written as a sum of distinct powers of $1+i$, then we obtain a fascinating snowflake spiral. This is the picture inside the square $[-30,30] \times [-30,30]$.



The property that $L(z) = 0$ if and only if $L(i-z) = i$ says that the black spiral and the white spiral are symmetric with respect to the point $i/2$.