Problem 12334
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Proposed by F. Stanescu (Romania).

Let f be a real-valued function on $[0,1]$ with a continuous second derivative. Assume that $f(0)=0$, $f'(0)=1$, $f''(0)\neq 0$, and $0<f'(x)<1$ for all $x\in(0,1]$. Let $x_1,x_2,\dots$ be a sequence with $0<x_1\leq 1$ and with

$$x_{n+1}=f\left(\frac{1}{n}\sum_{k=1}^{n}x_k\right)$$

for $n\geq 1$. Prove $\lim_{n\to\infty} x_n\ln(n)=-2/f''(0)$.

Solution proposed by Roberto Tauraso, University of Rome Tor Vergata, Rome, Italy.

Solution. We first note that for any $x\in(0,1]$, by the Mean Value Theorem, there is $t\in(0,x)\subset(0,1]$ such that

$$f(x)=f(x)-f(0)=f'(t)(x-0)=f'(t)x.$$

From $0<f'(t)<1$, it follows that $0<f(x)<x$, and therefore the sequence $(x_n)_n$ is contained in $(0,1]$. Moreover, by letting $z_n=\frac{1}{n}\sum_{k=1}^{n}x_k$, we have that $0<z_n\leq 1$, and

$$z_{n+1}=\frac{nz_n+x_{n+1}}{n+1}=\frac{nz_n+f(z_n)}{n+1}<\frac{nz_n+z_n}{n+1}=z_n.$$

Hence the decreasing and bounded sequence $(z_n)_n$ is convergent to a limit $L\in[0,1]$. By the Stolz-Cesàro Theorem and the continuity of f, we find

$$L=\lim_{n\to\infty} z_n=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}x_k\overset{\text{SC}}{=}\lim_{n\to\infty} x_{n+1}=\lim_{n\to\infty} f(z_n)=f(L).$$

Thus L is the only fixed point of f in $[0,1]$, i. e. $L=0$.
Since, by assumption, $f(x)=x+\frac{f''(0)}{2}x^2+o(x^2)$ as $x\to 0^+$, with $f''(0)\neq 0$, then

$$\begin{aligned}z_{n+1}&=\frac{nz_n+f(z_n)}{n+1}=\frac{nz_n+z_n+o(z_n)}{n+1}=z_n(1+o(1)),\\ x_n&=f(z_{n-1})=z_{n-1}(1+o(1))=z_n(1+o(1)).\end{aligned}$$

Finally, by the Stolz-Cesàro Theorem (note that $1/z_n$ is a strictly monotone and divergent sequence),

$$\begin{aligned}\lim_{n\to\infty} x_n\ln(n)&=\lim_{n\to\infty}\frac{x_n}{z_n}\cdot\lim_{n\to\infty}\frac{\ln(n)}{1/z_n}\overset{\text{SC}}{=}\lim_{n\to\infty}\frac{\ln(n+1)-\ln(n)}{\frac{1}{z_{n+1}}-\frac{1}{z_n}}\\ &=\lim_{n\to\infty}\frac{\ln(1+\frac{1}{n})z_{n+1}z_n}{z_n-z_{n+1}}=\lim_{n\to\infty}\frac{\frac{z_n^2}{n}(1+o(1))}{z_n-\frac{nz_n+f(z_n)}{n+1}}\\ &=\lim_{n\to\infty}\frac{z_n^2(1+o(1))}{z_n-f(z_n)}=\lim_{n\to\infty}\frac{z_n^2(1+o(1))}{-\frac{f''(0)}{2}z_n^2(1+o(1))}=-\frac{2}{f''(0)}.\end{aligned}$$

□